Speed of convergence in the Solow model

Solow model in per-efficiency unit form (ie \( y = \frac{Y}{eL}, k=\frac{K}{eL} \)):

\[ y = k^\alpha \]

Differential equation for capital:

\[ \dot{k} = sk^\alpha - (n + g + \delta)k \]

divide by \( k \) to get:

\[ \frac{\dot{k}}{k} = sk^{\alpha - 1} - (n + g + \delta) \]

We can graph the two parts of the right hand side of this equation.

[picture]

The difference between these two is \( \frac{\dot{k}}{k} \), the growth rate of the capital stock (per efficiency unit of labor). What about the growth rate of output? To see the relation between these, start with the production function, take logs and differentiate to get
So the growth rate of output is just proportional to the growth rate of capital.

So what happens to the growth rate of output per capita away from the steady state? If we are below the steady state, the growth rate of output monotonically decreases as we approach the steady state.

What happens to the growth of output if we increase saving. in this case, the $sk^{\alpha - 1}$ curve shifts up, we have a temporary increase in growth, then growth peters out to zero again.

We will now look more mathematically at the speed of output growth along the path to the steady state. We can re-write the production function as:

$$\ln(y) = \alpha \ln(k)$$

define $y^*$ and $k^*$ as the steady state levels of income and capital per efficiency unit.

$$\ln(y^*) = \alpha \ln(k^*)$$

so

$$\frac{1}{\alpha} [\ln(y) - \ln(y^*)] = \ln(k) - \ln(k^*)$$

(note that we have left time index out for convenience)

We know qualitatively what will happen when $y$ is above or below its steady state level. But what can we say quantitatively?

To make progress, we will linearize around the steady state.
It turns out that to do this, we will want to re-write the equation for output growth in terms of the log of capital, and linearize in that variable: this is called "log-linearizing." So first we write output growth as a function of the log of capital:

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} = \alpha (s k^{\alpha-1} - (n + g + \delta)) = \alpha s e^{(\alpha-1)\ln(k)} -\alpha (n + g + \delta) = f'(\ln(k))$$

To linearize, we have to pick a point around which to do the linearization. We pick the steady state, that is, ln($k^*$).

The linear approximation is:

$$\frac{\dot{y}}{y} \approx f'(\ln(k^*)) + f''(\ln(k^*))[\ln(k) - \ln(k^*)]$$

the derivative of $f'(\ln(k))$ is:

$$f' = \alpha s e^{(\alpha-1)\ln(k)} (\alpha - 1)$$

evaluating this at the steady state (when $\dot{k} / k = 0$), we can replace $s e^{(\alpha-1)\ln(k)}$ with $(n+g+\delta)$. so we get

$$\frac{\dot{y}}{y} = \alpha (\alpha - 1)(n + g + \delta)[\ln(k) - \ln(k^*)]$$

and we can replace the last part with the expression above to get:
\[
\frac{\dot{y}}{y} = (\alpha - 1)(n + g + \delta)\left[\ln(y) - \ln(y^*)\right] = \gamma\left[\ln(y^*) - \ln(y)\right]
\]

Where \( \gamma = (1 - \alpha)(n + g + \delta) \)

we can re-write this as:

\[
\frac{d\ln(y)}{dt} = \gamma\left[\ln(y^*) - \ln(y)\right]
\]

which says that the log difference between the current level of output per efficiency unit and the steady state level decays exponentially. Another way to write this is

1 To do this step we solve a differential equation as follows. For convenience, define \( x = \ln(y) \). So the differential equation becomes:

\[
\dot{x} = -\gamma [x - x^*]
\]

re-arrange:

\[
\dot{x} + \gamma x = \gamma x^*
\]

multiply both sides by \( e^{\gamma t} \) and integrate:

\[
\int e^{\gamma t} [\dot{x} + \gamma x]dt = \int e^{\gamma t} \gamma x^* dt
\]

\[
e^{\gamma t} x = e^{\gamma t} x^* + b
\]

where \( b \) is a constant of integration, which must be determined. We can re-arrange to get.

\[
x = x^* + e^{-\gamma t} b
\]

Now we find the value of \( b \) from the starting condition: \( x(0) = x_0 \).

\[
x(0) = x_0 = x^* + b \quad \Rightarrow \quad b = x_0 - x^*
\]
If we want to know how long it takes the economy to close half of the gap in log income, we just set

\[ \exp(-\gamma t) = \frac{1}{2} \]

so

\[ t = \frac{\ln(2)}{\gamma} \approx \frac{.7}{\gamma} \]

so for \( \alpha=1/3, \ n=.01, \ g+\delta=.05 \), this says that the "half life" of the log difference should be about 17 years. If \( \alpha = 2/3 \), it should be about 35 years.

(note that this is the half life of the log difference -- so it is like the time required to go from 1/4 of the steady state to 1/2 of the steady state). It is not the half life of the absolute difference.

Note that things like \( s \), which determine the level of the steady state, do not determine the speed at which the country approaches the steady state (this is true because of Cobb-Douglas production).

--------------------------------------------

**Extending the Solow Model to Include Human Capital**

The dynamics in the Solow model come from the accumulation of capital. We have been thinking of capital as buildings and machines. The return to capital of this sort is interest and profits (In the real world, some of profit is also a return for entrepreneurship, but we ignore this issue).

One of the facts that is consistent with the Cobb-Douglas production function is the constancy of capital's earnings as a share of output. This is roughly 1/3.

so

\[ x = x^* + e^{\gamma t} (x_0 - x^*) = (1 - e^{-\gamma t}) x^* + e^{\gamma t} x_0 \]
But now let's think about a broader definition of capital. Not just physical stuff, but also human capital. Invest and earn a return (higher wages) just like physical capital.

Most frequently run regression in economics: Human capital earnings function [graph].

What this says is that a large fraction of wages are returns to human capital rather than return to raw labor.

Thus, saying capital's share in income is 1/3 is wrong, since some of what we have been calling labor's share is really return to capital.

So let's think about a production function with two different kinds of capital, physical and human. For convenience we assume Cobb-Douglas:

\[
Y = K^\alpha H^\beta (eL)^{1-\alpha-\beta} \quad \alpha + \beta < 1
\]

We will do our analysis in per-efficiency unit terms, so \( h = H/(eL) \), etc. So the production function is:

\[
y = k^\alpha h^\beta
\]

We will assume that human capital and physical capital depreciate at the same rate, \( \delta \). Let \( s_k \) be the fraction of output that is used to produce physical capital, and \( s_h \) be the fraction of output that is used to produce human capital. The accumulation equations for physical and human capital are:

\[
k' = s_k k^{\alpha} h^\beta - (n + g + \delta)k
\]

\[
h' = s_h k^\alpha h^{\beta} - (n + g + \delta)h
\]
To find the steady state, we set both of these to zero and solve:

\[
k^* = \left( \frac{S_k}{S_h} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left( \frac{n + g + \delta}{S_k} \right)
\]

\[
h^* = \left( \frac{S_k}{S_h} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left( \frac{n + g + \delta}{S_h} \right)
\]

Now we can look at the speed of convergence to the steady state in this version of the model:

The equations for factor accumulation can be re-written as:

\[
\frac{\dot{k}}{k} = s_k k^{\alpha-1} h^{\beta} -(n + g + \delta)
\]

\[
\frac{\dot{h}}{h} = s_h k^{\alpha} h^{\beta-1} -(n + g + \delta)
\]

Starting with the production function, we can take logs and differentiate with respect to time to get:

\[
\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + \beta \frac{\dot{h}}{h}
\]

so we can write
The linear approximation for this case is

\[
\frac{\dot{y}}{y} = \alpha_{sk} k^{\alpha-1} h^\beta + \beta_{sh} k^{\alpha} h^{\beta-1} - (n+g+\delta)(\alpha+\beta)
\]

\[= f(\ln(k), \ln(h))\]

The linear approximation for this case is

\[
f(\ln(k), \ln(y)) \approx f(\ln(k^*), \ln(h^*)) + f_1'(\ln(k^*), \ln(h^*))[\ln(k) - \ln(k^*)] + f_2'(\ln(k^*), \ln(h^*))[\ln(h) - \ln(h^*)]
\]

where \(f_1\) indicates the derivative with respect to the first argument (note that the derivative is being taken w.r.t. \(\ln(k)\), not w.r.t. \(k\)).

\[
f_1 = \alpha_{sk} e^{(\alpha-1)\ln(k)} e^{\beta\ln(h)}(\alpha - 1) + \beta_{sh} e^{\alpha \ln(k)} e^{(\beta-1)\ln(h)}\alpha
\]

We are evaluating this at the steady state, \(\frac{\dot{k}}{k} = \frac{\dot{h}}{h} = 0\); so we can substitute from the equations of motion to get

\[
f_1'(\ln(k^*), \ln(h^*)) = \alpha(n+g+\delta)(\alpha - 1) + \beta(n+g+\delta)\alpha = \alpha(\alpha + \beta - 1)(n+g+\delta)
\]
and by symmetry we will have a similar thing for the derivative of \( f \) with respect to \( \ln(h) \). So

\[
\frac{\dot{y}}{y} \approx \alpha (\alpha + \beta - 1) (n + g + \delta) [\ln(k) - \ln(k^*)] + \beta (\alpha + \beta - 1) (n + g + \delta) [\ln(h) - \ln(h^*)]
\]

putting these into the linearization formula (and noting that \( f(\ln(k^*), \ln(h^*)) = 0 \)), we get

\[
\frac{\dot{y}}{y} = (\alpha + \beta - 1) (n + g + \delta) [\alpha (\ln(k) - \ln(k^*)) + \beta (\ln(h) - \ln(h^*))]
\]

but from the production function we have

\[
\ln(y) = \alpha \ln(k) + \beta \ln(h)
\]

so we can substitute for the term in brackets

\[
\frac{\dot{y}}{y} \approx (1 - \alpha - \beta) (n + g + \delta) [\ln(y^*) - \ln(y)]
\]

Notice that this looks a lot like what we had before, except now the speed of convergence is slower, corresponding to the higher level of capital-like stuff in the economy.

**Capital Mobility in a Solow Model**

Now we do the model again, assuming that physical capital flows internationally to equalize the marginal produce less depreciation to some world rate \( r \). \( r \) is exogenous. The assumption that whatever happens in our country has no effect on the world interest rate is usually called the assumption of a “small open economy”

First consider the Solow model with physical capital as the only factor of production.
Because the economy is open, we know that \( f'(k) - \delta = r \). This immediately pins down the level of the capital stock and the level of output. For example, for Cobb-Douglas production:

\[
r = f'(k) - \delta = \alpha k^{\alpha-1} - \delta
\]

\[
k = \left[ \frac{\alpha}{(r+\delta)} \right]^{1/(1-\alpha)}
\]

Notice that the saving rate doesn't matter at all to determining the level of output!

Now consider the model with both physical and human capital. Obviously, if both human and physical capital flow between countries, then this will look just like the one-factor model just presented. In particular, the country will jump to its steady state immediately. Instead, we assume that while physical capital can flow between countries, human capital cannot. (The model is that of Barro, Mankiw, and Sala-i-Martin; except that they look at a fully optimizing model (like Ramsey), while I will do the "Solow" version).

\[
f_k - \delta = r
\]

from this, we can solve for the level of capital as a function of the level of human capital.

\[
\alpha k^{\alpha-1} h^\beta = r + \delta
\]

\[
k = \left( \frac{\alpha}{r+\delta} \right)^{\frac{1}{1-\alpha}} \frac{\beta}{h^{\frac{\beta}{1-\alpha}}}
\]

So we can solve for \( y \) as a function of just \( h \):

\[
y = h^\beta k^\alpha = h^{\frac{\beta}{1-\alpha}} \left( \frac{\alpha}{r+\delta} \right)^{\frac{\alpha}{1-\alpha}}
\]
How does human capital accumulate? There is a potential problem here, since $y$ is gross domestic product, but not gross national product (since capital may have flowed into or out of the country). One might think that human capital investment should be a constant fraction of GNP rather than GDP. On the other hand, we could argue that payments to human capital are a constant fraction of GDP (not GNP) because of cobb-douglass production. So if accumulation of $h$ was financed by a constant fraction of payments to $h$ (or payments to raw labor), then assuming it to be constant fraction of GDP would be OK. In any case, this is what we assume:

$$\dot{h} = s_h y - (n + g + \delta) h$$

To get from here to the solution, there is a shortcut that we can take: the production function, once $k$ is eliminated, looks just like the production function in the single factor case (where $k$ was the factor) along with a constant in front. The only difference is that instead of the exponent being $\alpha$, it is $\beta/(1-\alpha)$. We can just replace this for $\alpha$ in the solution and immediately get the answer.

In the case where the production function was $y = k^\alpha$, the solution was

$$\frac{\dot{y}}{y} \approx (\alpha - 1)(n + g + \delta)[\ln(y) - \ln(y^*)]$$

so substituting we get

$$\frac{\dot{y}}{y} \approx \frac{(n + g + \delta)(1-\alpha-\beta)}{1-\alpha}(\ln(y^*) - \ln(y))$$
Which we expect to be between the speeds of convergence in the last two models.

**Growth Regressions -- The Basic Approach**

**Production Function**

- written as Cobb-Douglas for convenience (this is only sometimes necessary). We suppress human capital for now.

\[
Y_i = K_i^\alpha \left( A_i L_i \right)^{1-\alpha}
\]

Where \( A_i \) is a country-specific productivity term.

In per efficiency unit of labor terms (ie \( y = Y / AL \)) terms:

\[
y = k^\alpha
\]

capital accumulation equation:

\[
k = iy - (n + g + \delta)k
\]

where \( i \) is the fraction of output invested, \( n \) is the growth rate of the labor force, \( \delta \) is the rate of depreciation, and \( g \) is the growth rate of \( A \).

These two equations can be solved for the steady state level of output per efficiency unit of labor:
turning this back into per capita terms, and taking logs, we get

\[ \ln(Y/L)_{i,ss} = \ln(\bar{A}_i) + \frac{\alpha}{1-\alpha} \ln(i_{i,i}) - \frac{\alpha}{1-\alpha} \ln(n + g + \delta) \]

**Approach #1: Cross Sectional Levels Regression**

To treat this cross sectional “levels” regression, we have to deal with two complications:

1. Countries not at their steady state

   We assume that deviations from steady state are mean zero in logs

   \[ \ln(Y/L)_i = \ln(Y/L)_{ss,i} + \epsilon_{ss,i} \]

2. Countries have different levels of productivity

   We assume

   \[ \ln(A_i) = \ln(\bar{A}) + \epsilon_{A,i} \]
So we estimate

\[ \ln(Y/L)_i = \gamma_0 + \gamma_1 \ln(i_i) + \gamma_2 \ln(n_i + g + \delta) + \varepsilon_i \]

(Estimated with or without the restriction that \(\gamma_1 = -\gamma_2\))

Key assumptions:

-- error term is uncorrelated with RHS variables. Specifically:

-- there is no effect of high income on investment rate or other rates of accumulation

-- there is no correlation between the random shock to productivity and rates of accumulation

-- Values of the production function parameters (\(\alpha\)) are constant across countries.

Finding:

Implied value of \(\alpha\) is too large, in comparison with capital’s share of income. Likely cause is omitted variable bias, since human capital accumulation is probably correlated with physical capital accumulation.

Solution: measure the rate of investment in human capital \(i_h\) in addition to the rate of investment in physical capital. We assume that human capital depreciates at the same rate as physical capital.

The production function is

\[ Y_i = K_i^\alpha H_i^\beta (A_i L_i)^{1-\alpha-\beta} \]
The equation for the accumulation of human capital (in per efficiency unit terms) is

\[ h = i_h y - (n + \delta + g)h \]

The equation for the steady state is now

\[ \ln(Y/L)_{i,ss} = \ln(A_i) + \frac{\alpha}{1 - \alpha - \beta} \ln(i_k, i) + \frac{\beta}{1 - \alpha - \beta} \ln(i_{ki}) - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n_i + g + \delta) \]

So we estimate

\[ \ln(Y/L)_i = \gamma_0 + \gamma_1 \ln(i_k, i) + \gamma_2 \ln(i_{ki}) + \gamma_3 \ln(n_i + g + \delta) + \varepsilon_i \]

where, once again, we can impose and test a restriction, namely that

\[ \gamma_1 + \gamma_2 = -\gamma_3 \]

If this restriction is imposed, then we can use the values of gamma to back out beta and alpha.

[Note about how the restriction is imposed. You can do this either by changing the RHS variables, as in MRW, or by estimating the thing by NLLS. The advantage of the latter is that you don=t have to use the delta method or something to figure out the standard errors, which MRW did.]

■ explains a lot of the variance of the cross section
value of $\alpha$ implied by $\beta$ are not bad in the sense that the alpha looks close to what we think that capital share of income should be, and the value of beta implies that half of labor earnings are really return to human capital.

**Approach #2: “Growth Regressions”**

One objection to levels regressions is that maybe countries are not distributed evenly around their steady states -- for example, it may be that most countries are below their steady states. A second objection is that, if the world is characterized by Endogenous Growth (of a crude form) there is not steady state of relative income. For these reasons we might want to look at income growth, rather than its level, as the dependent variable. [On the other hand, Hall and Jones make a good argument that over the time period for which we have data, there is not that much that can be learned by looking at growth rate. When we look at levels, we are implicitly looking at growth rates over a very long period, and this is more useful data.]

[Exercise to think about: suppose that the investment rate follows an AR(1). Then the investment rate will be (negatively) correlated with the distance from the steady state.]

Going back to the Solow model, it can be shown that the rate of growth of output per efficiency unit of labor is proportional to the gap (in logs) between the current level of output and the steady state level

$$\frac{\dot{y}}{y} = \lambda (\ln(y_{xx,i}) - \ln(y_{i}))$$

Where above we showed that

$$\lambda \approx (1 - \alpha - \beta)(n + g + \delta)$$

However, it will turn out that this relation between lambda and the other parameters ends up not being used for anything (why not? I don’t really know.)
Substituting the equation for steady state into the equation for convergence speed, we get

\[
\frac{\dot{y}}{y} = \lambda \left( \ln (A_i) + \frac{\alpha}{1 - \alpha - \beta} \ln (i_{ki}) + \frac{\beta}{1 - \alpha - \beta} \ln (i_{hi}) - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln (n_i + g + \delta) - \ln (y_i) \right)
\]

This allows us to relax one of the assumptions above, namely that countries are randomly distributed around their steady states. We can regress

\[
growth = \gamma_0 + \gamma_1 \ln \left( \frac{Y}{L} \right)_i + \gamma_2 \ln (i_{ki}) + \gamma_3 \ln (i_{hi}) - \gamma_4 \ln (n_i + g + \delta) + \epsilon_i
\]

This regression can be estimated on a cross-section of growth rates

- notice that we have to add an assumption that *growth rates* of A are distributed randomly and not correlated with RHS variables.

Key finding: coefficient $\gamma_1$ is less than zero. This is the notorious “conditional convergence” result. The coefficient $\gamma_1$ is a direct estimate of the parameter lambda (with the sign reversed). So we can interpret this coefficient directly as the speed of convergence toward the steady state. This is important, since one of the issues with which we will wrestle over and over is how far countries are from their steady states.

The estimates of $\lambda$ tend to be around .02. Barro and Sala-i-Martin claim that this is a universal result (not clear if they are serious about this.)
Notice that this accords well with our \textit{a priori} view, since we derived in the linearization that

$$\lambda \approx (1 - \alpha - \beta)(n + g + \delta)$$

and if we use parameters of alpha = beta = 1/3 and n approx .01, g+delta approx .05, then we get .02.

[notice also that we are cheating a little here, since if countries have different n then they should have different speeds of convergence to their steady states, but we ignore this and estimate a single $\lambda$ for all countries.]

**Implications for Endogenous Growth**

Endogenous growth models of the “Ak” variety feature constant returns to factors of production that can be accumulated. This can be seen as the limiting case of starting with a production function in which there is an exponent alpha on capital, and then taking alpha to one. Looking at the speed of convergence calculations we have done, if alpha is one, then the half-life of log output differences is zero and the catch up term lambda is zero (that is, the formula for $\lambda$ is $\lambda = (n + g + \delta)(1 - \alpha)$, in the case where there is only physical capital).

So a key test of endogenous growth is the coefficient on initial output in the growth regression, since that is a direct estimate of lambda. Endogenous growth says it should be zero. So the estimate of convergence speed of 2% is a big defeat for this type of model.

However, this is only a critique of “old style” endogenous growth models. Newer classes of models assume that countries in the sample have the same levels (or similar levels) of technology, and that the growth of this technology is endogenous. Holding technology fixed (which is done in a cross-country regression), these models would still feature decreasing returns to accumulatable factors, and thus would show conditional convergence.

**Other Coefficients**
Regarding the other coefficients, there is once again a linear constraint that we can impose and test. Once we have imposed this constraint, we can also use the estimated values of $\gamma_1$, $\gamma_2$, and $\gamma_3$ to back out the implied values of alpha and beta (to do this we divide the second two by the first, and this makes the lambda go away.)

Issues with this whole “growth regression” approach: All of this is plagued by omitted variables and reverse causality. We see that rates of accumulation ($s_k$, $s_h$, and $n$) differ among countries. But these are not exogenous. In particular, you might worry that they are correlated with the level or growth rate of $A$. This will of course bias the estimates. As we will see in a few lectures, there is a different direction we can go, using development accounting, where we can learn about the level and growth rate of $A$ to see whether they are indeed correlated with other stuff. However, to go that way, we cannot estimate the parameters of the production function (alpha and beta). Rather, we have to take them as given.

---

### Adding Other Stuff to the Right Hand Side of Growth Regressions

One of the most common uses for the “growth regression” is to look at other stuff

Justification: suppose that the level of productivity $A$ is affected by some characteristic.

\[
\ln(A_i) = A_0 + \psi B_i + \epsilon_{A_i}
\]

where $B$ is some measurable aspect of country $i$ and $\psi$ is some unknown coefficient.

Examples of $B$:

- openness
- the Africa dummy
democracy

e tc.

We can just add B to the right hand side of the growth regression. How do we interpret its coefficient?

In words: the coefficient on B says that holding the level of income constant and the other things constant, a country with B=1 would grow gamma amount faster than a country that had B=0.

This is often interpreted as “B will lead to growth that is xx higher or “B raises growth by xx.”

Why this is wrong: the increase in growth is estimated to take place only holding other things constant. But higher growth will raise the level of income, which is one of the things that was being held constant. And of course, since this is a model with convergence, there is a negative coefficient on the level of income. So eventually, the higher growth will have raised income enough to set back growth to the level it would have been if B were not raised.

To undo this, we would have to think: how much bigger will steady state income have to be to offset this.

In math: go back to the where B*psi would show up: we can see that in the growth regression, to recover the coefficient psi, we have to divide by the convergence rate (see algebra below). Doing this, we can convert the estimate of gamma into a measure of psi. Also, from the way that we have put productivity into the production function, this will be the amount by which the steady state differs.

Going back to the equation for growth and substituting in the equation for A_i, we get

\[
\frac{\dot{y}}{y} = \lambda \left( B_0 + \psi B + \frac{\alpha}{1 - \alpha - \beta} \ln(i_{ki}) + \frac{\beta}{1 - \alpha - \beta} \ln(i_{hi}) - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n_i + g + \delta) - \ln(y_i) \right)
\]

So if we run a regression of growth on B and on the usual other stuff, we get
\[ \text{growth} = \gamma_0 + \gamma_1 \ln \left( \frac{Y}{L} \right)_i + \gamma_2 B + \ldots + \varepsilon_i \]

Mapping this back into the parameters that we are interested in:

\[ \gamma_1 = -\lambda \]
\[ \gamma_2 = \lambda \psi \]

So to get an estimate of the effect of B on productivity (A), we have to divide \(\gamma_2\) by negative \(\gamma_1\).

Question: suppose that the “other stuff” did not affect productivity, but rather affected (say) rates of accumulation (ie we measured culture, and culture lowered the saving rate, etc.). The answer is that there would be no effect on the coefficient on “other stuff,” since saving is already in the regression. So it is perfectly possible to think that culture is a determinant of output, but that it doesn’t enter the standard growth regression. In other words, the only way that it _will_ enter the regression is if it affects productivity.

One big point of this specification is that you have to be careful about saying that "doing X causes growth." The interpretation that I like is that this raises your steady state level of output, and so, conditional on your initial level, raises the growth rate. The other interpretation is that doing X raises your growth rate permanently. How to differentiate? I think it is just a matter of looking at the coefficient on the initial level of output -- if this is negative, then it must be the case the doing X gets you growth only temporarily.

A famous example:


The disease measure that they use is the product of the fraction of a country’s population living in regions with high malaria risk in 1965 times the fraction of malaria cases that were due to \(P. falciparum\). The index
is in the range of 0.75-1.00 for most of sub Saharan Africa. The dependent variable is the average rate of income growth over the period 1965-1990. Additional controls include measures of geography, quality of institutions, schooling, and life expectancy (to control for other diseases).

Gallup and Sachs’s estimate for the coefficient $\gamma_1$ is -1.3%. The interpretation is that ceteris paribus a country with a malaria index of 1.0 would grow 1.3% per year more slowly than a country with an index of zero – or, alternatively, that wiping out malaria in a country where the index was 1.0 would raise the growth rate of income by 1.3% per year. The conclusion that malaria has this large an effect on growth was famously enshrined in the Abuja declaration of 2000, signed by 57 African heads of state.

An alternative way of presenting the Gallup and Sachs result is to use their estimate of the conditional convergence parameter, $\gamma_2$, which is -2.6. Dividing the estimate of $\gamma_1$ by that of $\gamma_2$ (and changing the sign) yields an estimate for $\beta_1$ of 0.5. The interpretation is that going from a malaria index of one to a malaria index of zero would raise the log of steady state income per capita by 0.5, implying that the level of steady state income would rise by 65%. Expressed this way, the Gallup and Sachs result is somewhat easier to compare to other measures of the role that disease plays in affecting economic growth.

Endogenous Growth Models

Growth regressions of the Barro or MRW style show us that first generation endogenous growth models of the Ak style are not sensible. However, we are still interested in more sensible models where it is technology rather than factor accumulation that drives the endogenous growth.

Lucas Growth Model

The Lucas article “On the Mechanics of Economic Growth” goes through a model of growth with human capital.

production fn: $Y = K^\alpha (u^*h^*L)^{1-\alpha}$

Where L is labor, h is the amount of human capital per person, and u is the fraction of their time that people spend working.
or in per capita terms

\[ y = k^\alpha (uh)^{(1-\alpha)} \]

So production is CRS in capital and this quality and time adjusted labor input.

capital has the usual differential equation: 
\[ \dot{k} = y - c - (n + \delta)k \]

But now there is also a production function for human capital. To produce human capital, you have to take time away from producing goods. So \(1-u\) is the fraction of work time spent producing human capital,

\[ \dot{h} = \phi h(1-u) \text{ or alternatively } \frac{\dot{h}}{h} = \phi(1-u) \]

This says that human capital is produced by a constant returns to scale production function, where \( \phi \) is just some constant. That is, if we double the amount of input into producing human capital \((h^*(1-u))\), then we double the amount of human capital produced.

[notice that we treat human capital as not being depreciating, and as not being diluted by the arrival of new people. Thus in this formulation, human capital is really like knowledge; and creation of new human capital is really like research and development. Allowing for a \(-(n+\delta)h\) term in the equation would not really change things that much.

Lucas goes on to solve this as a social planner's problem using a Hamiltonian. There are two state variables \((h \text{ and } k)\) and two control variables \((u \text{ and } c)\).

For simplicity, we will do the “Solow” version of the model, in which \(u\) and \(s(=(y-c)/y)\) are taken to be exogenous. Recall that the steady state of the Ramsey model looks just like the steady state of the Solow model in terms of output and saving being constant, and output being a function of the saving rate (the only difference between the two models being that for the Ramsey, we can find the saving rate as an explicit function of the taste parameters). Here, the analogue of the steady state is the “balanced growth path” (see below). The balanced growth path of the Lucas model looks just like the balanced growth path of the model presented here: \(u\) and \(s\) are constant, and the growth rate of output is a function of them. The only differences
between the true Lucas model and the one presented here are in the dynamics off of the balanced growth path, and that in the Lucas model u and s are explicit functions of the taste parameters.

re-write capital accumulation using s instead of c:

\[ \dot{k} = s y - (n+\delta)k \]

so

\[ \frac{\dot{k}}{k} = s \frac{y}{k} - (n + \delta) = s k^{1-\alpha} (uh)^{\alpha-1} - (n + \delta) = s \left( \frac{k}{h} \right)^{\alpha-1} u^{\alpha-1} - (n + \delta) \]

Now we draw a diagram with both \( \dot{k} \) and \( \dot{h} \) on the vertical axis and \( k/h \) on the horizontal axis. The \( \dot{h} \) equation is just a horizontal line. The \( \dot{k}/k \) equation slopes downward.

What happens to \( k/h \) over time? Where the two curves cross, \( k \) and \( h \) are growing at the same rate, and so \( k/h \) will remain constant. Call this \( (k/h)^* \), the balanced growth ratio of \( k \) to \( h \). For lower levels of \( k/h \), \( k \) will be growing faster than \( h \), so \( k/h \) will be growing over time. For \( k/h \) bigger, vice versa. So whatever level of \( k/h \) you start with, eventually it will move to \( (k/h)^* \).

What about the growth rate of output? Starting with the production function, we can take logs and differentiate:

\[ \ln(y) = \alpha \ln(k) + (1-\alpha) \ln(u) + (1-\alpha) \ln(h) \]

\[ \frac{\dot{y}}{y} = \frac{\dot{k}}{k} + (1-\alpha) \frac{\dot{h}}{h} \]

So in the long run (along the balanced growth path), the growth rates of output, capital, and human capital are all equal, and are all determined by the \( \dot{h}/h \) equation. The lower is \( u \), the faster is growth along the balanced growth path.
Unlike the Ak model, there are transitional dynamics, which depend on the initial k/h ratio: if k/h is initially lower than the balanced growth level, then output will initially grow faster than it will in the long run.

Do the experiment of lowering u from the steady state. What do time series look like? The level of y initially falls, but the growth rate jumps (although we don’t know whether it jumps up or down), then continues to rise.

**Knowledge spillovers in the Lucas setup.**

Consider the following two-country version of the Lucas growth model.

\[ y_i = \text{output per capita in country } i \quad (i = 1, 2) \]

\[ k_i = \text{capital per capita in country } i \]

\[ n = \text{population growth} \quad (\text{assumed to be the same in both countries}) \]

\[ \delta = \text{depreciation} \quad (\text{assumed to be the same in both countries}) \]

\[ s = \text{saving rate} \quad (\text{same in both countries}) \]

\[ u_i = \text{fraction of their time that people spend working in country } i \]

\[ (1-u_i) = \text{fraction of their time that people spend building human capital in country } i \]

Production:

\[ y_i = k_i^\alpha (u_i h_i)^{(1-\alpha)} \]

Capital accumulation:

\[ \dot{k}_i = sy_i - (n + \delta)k_i \]
So far all of this is like the model presented in class. For human capital accumulation, we add a new assumption. Let $h_l$ be the level of human capital per capita in the country that has a higher level of human capital (the "leader"), and $h_f$ be the level in the country that has less human capital (the "follower.")

We assume that human capital production in the leader country works exactly as in Lucas' model:

$$\dot{h}_l = \phi (1-u_l)h_l$$

Human capital is produced in the follower country by two methods: first, there is the same production as in the leader country. But second, human capital (which in this model is the same as "knowledge") spills over from the leader country to the follower country. The amount of this transfer depends on the difference in their levels of knowledge.

$$\dot{h}_f = \phi (1-u_f)h_f + \beta(h_l - h_f) \quad \beta > 0$$

Assume that the two countries have the same saving rate, but $u_1 < u_2$. In other words, in country 2 people spend a larger fraction of their time working (and a smaller fraction producing human capital) than in country 1.

A) Describe the steady state of the model. Solve for each country's growth rate in steady state.

Obviously country 2 will be the technology follower. So we can solve for country 1 exactly as if it were alone in the world. Now the key question is what do the country=s growth rates of $h$ look like? Answer: they must be the same: if not, then either country 2 would catch up or fall infinitely far behind.

(Figure: put $h_1 / h_2$ on the horizontal axis, and the two $h$ dot lines on the vertical)
B) Solve for the relative level of human capital per person in the two countries in steady state. How do the parameters phi and ß affect the ratio of $h_1/h_2$.

Set the two $\dot{h}/h$ equations equal. Then ..... 

C) (hard) Solve for the relative level of consumption in the two countries in steady state. How (and why) do the parameters phi and ß affect the relative level of consumption.

Extending this idea:

Basu, Feyrer, and Weil. Instead of doing the “Solow” version of this model, we can do the “Ramsey” version, in which we take u as a decision variable. (For simplicity we don’t think about changing s; in fact, in the paper, we take s as given.)

First, think about the one-country case. Suppose that there is a social planner who cares about discounted consumption. Then clearly if he has a higher value of $\theta$, there will be less R&D, slower growth, but higher initial consumption.

Now, consider the case of a world with two countries. Suppose, for simplicity, that R&D in one country is fixed. Suppose that I am a social planner, and I was considering doing exactly the same amount of R&D as the other guy is already doing. My optimal path will now be to do _less_, and let him be the leader.

It is even possible that I might have wanted to grow faster than him, but if his is fixed, I will let him be the leader.

(All of this gets into dynamic games, which are too hard for me.... )
[Acemoglu, Robinson, and Verdier (2012) do a model sort of like this, but with many other
features regarding income distribution.]

Problems

1) [Midterm exam, 2008] A country is described by the Solow model with production function

\[ Y = K^\alpha (eL)^{1-\alpha} \]

Population grows at rate \( n \), capital depreciates at rate \( \delta \), \( e \) grows at rate \( g \), and the saving rate is \( s \). The economy is in steady state. Suddenly, the growth rate of population increases by some amount \( \Delta n \). By how much does the growth rate of total output increase or decrease instantaneously?

2) Country one is exactly described by the standard Solow model (with no technological progress):

\[ y = k^\alpha \]

\[ \dot{k} = sy - (n + \delta)k \]

The values of the parameters are: \( \alpha = .5, \quad (n+\delta) = .1, \quad s = .1 \)

Country 2 is the same as country 1, except that the saving rate is a function of the capital stock:

\[ s = s_0 \left( \frac{1}{1+k} \right) \]

Where \( s_0 = .2 \).
A. Show that the two countries have the same steady state levels of output. Intuitively, which country should move more rapidly toward its steady state?

B. Linearize around the steady state to get an expression for output growth of the form:

\[ \frac{\dot{y}}{y} = -\gamma (\ln(y) - \ln(y^*)) \]

How does the value of \( \gamma \) compare in the two cases?

3) (Final exam, 2008) A country is described by the version of the Lucas model presented in class:

\[ y = k^\alpha (uh)^{(1-u)} \]

\[ \frac{\dot{h}}{h} = \phi(1-u) \]

\[ \dot{k} = sy - \delta k \]

The country is on a balanced growth path. Now suppose that there is an increase in \( u \). For convenience, consider an infinitesimal increase in \( u \), so you can just take derivatives.

What are the conditions regarding the parameters \( u, \phi, s, \delta, \) and \( \alpha \) such that the growth rate of output immediately following the rise in \( u \) will be the same as it was along the old balanced growth path?
4) Consider a variation on the Lucas model analyzed in class.

Production (in per capita terms) is

\[ y = k^{0.5} h^{0.5} \]

where \( k \) and \( h \) are per capita levels of physical and human capital.

Physical capital is accumulated according to

\[ \dot{k} = s_k y - \delta k \]

Human capital is accumulated according to

\[ \dot{h} = s_h y - \delta h \]

Further, it just so happens that \( s_k = s_h = \delta \)

A. What is the steady state growth rate of output?

B. Suppose that a country is in steady state, when suddenly 3/4 of its physical capital (but none of its human capital) is destroyed. Will the growth rate of output immediately after the shock be higher, lower, or the same as the growth rate in steady state?
5) Three countries, A, B, and C, are all described by the version of the Lucas model presented in class:

\[ y = k^\alpha (uh)^{(1-\alpha)} \]

\[ \dot{h} / h = \phi^* (1-u) \]

\[ \dot{k} = sy - (n+\delta)k \]

Initially, all three countries have the same levels of s, u, h, and k. In year \( t \), dictators seize power in countries A and B and engage in policy experiments. Country A raises the rate of saving to some higher level, holding u constant. Country B reduces u to some lower level, holding s constant. Country C keeps both s and u constant. After 10 years, the dictators are overthrown, and s and u are returned to their original levels in countries A and B.

Draw a graph showing the time paths of the growth rates of output in the three countries during and after the period of policy experimentation. Distinguish between different special cases based on the parameters, if appropriate. Draw a second graph showing the time paths of the log of output in the three countries.

6) [midterm exam, 2004] Consider a variation of the Lucas growth model. The production function is

\[ y = (u_k k)^{\alpha} (u_h h)^{(1-\alpha)} \]

Where \( u_k \) is the fraction of physical capital that is used in producing output and similarly \( u_h \) is the fraction of human capital that is used in producing output.
The accumulation equations for $k$ and $h$ are

$$\dot{k} = \phi (1-u_k)k - (n + \delta) k$$

$$\dot{h} = \phi (1-u_h)h - (n + \delta) h$$

A) solve for the steady state growth rate of output

B) Suppose that $\alpha= .5$, $u_k = u_h = .5$, $(n + \delta = .05)$, $\phi = 0.1$. A country is in steady state.
Now suppose that $u_k$ falls to .125, while $u_h$ remains constant. How many years (approximately) will it take until output regains the level it would have had if there had been no change is $u_k$?
[Helpful hint: $\ln(2) \approx 0.70$].

7) [core exam, 2009] Consider the following simplified version of the two-country Lucas growth model with technology spillovers:

$y_i =$ output per capita in country $i$ \ ($i = 1, 2$)

$u_i =$ fraction of their time that people spend working in country $i$

$(1-u_i) =$fraction of their time that people spend building human capital in country $i$

Production:

$$y_i = u_i h_i$$
Consumption is equal to output. There is no population growth. There is no capital in production.

Human capital accumulation in the lead country is given by

\[ \dot{h}_l = \phi (1-u_l)h_l \]

Human capital accumulation in the follower country

\[ \dot{h}_f = \phi (1-u_f)h_f + \beta (h_l - h_f) \quad \beta > 0 \]

In country 1, the fraction of their time that people spend working is fixed at \( u_1 \).

Consider only the case where \( u_2 \geq u_1 \), so that country 2 is not the technology leader.

Assume that the countries are along a balanced growth path in which the ratio of technologies is constant.

A. Calculate income in country 2 as a function of \( u_2, u_1, \) and \( h_1 \).

B. Calculate the derivative of the answer to part A as a function of \( u_2 \).

C. Looking at your answer to part B, discuss the following. Suppose that \( u_2 \) is chosen by a social planner in country 2 who has standard intertemporal preferences and who takes the value of \( u_1 \) as fixed. What values of \( u_2 \) can be ruled out as possible optima chosen by that social planner along a balanced growth path? How do these depend on the parameters \( \phi \) and \( \beta \)? Explain both in terms of the equation and in terms of the underlying economics of the problem.

8) (core exam: 1996) Consider the following hybrid of the Solow model and the Lucas model with human capital.
Output is produced with physical capital, human capital, and labor according to the production function (written in per-worker terms):

\[ y = k^\alpha [(1-u)h]^{1-\alpha} \]

where \( y \) is output per worker, \( h \) is human capital per worker, \( k \) is physical capital per worker, and \((1-u)\) is the fraction of their time that workers spend producing output.

There is no population growth. Physical capital accumulates according to

\[ \dot{k} = sy - \delta k \]

where the saving rate, \( s \), is exogenous and fixed. When workers are not producing output, they are producing human capital (there is no leisure). The only input required to produce human capital is time (unlike the Lucas model, where human capital itself is one of the inputs to human capital production). Human capital also depreciates at the same rate (\( \delta \)) as physical capital. Thus, the evolution of human capital per capita is given by:

\[ \dot{h} = u - \delta h \]

Assume that \( s = \delta \).

A. Describe the steady state of the model. Is it one with a constant level of output or with a constant growth rate of output?

B. Solve for the level of \( u \) that maximizes either the level or the growth rate of output (depending on which is constant in steady state).
C. Let \( u^* \) be the level of \( u \) that you solved for in part B. Suppose that the economy is in steady state with \( u < u^* \). Suddenly, \( u \) jumps up to \( u^* \). Use a phase diagram in \((h,k)\) space to analyze the behavior of \( h \) and \( k \). Draw time series pictures showing how \( h, k \) behave during the transition to the new steady state.