Four Lectures on Cooperative Game Theory

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1 Lecture 1: Introduction and Cooperative Bargaining

The literature on cooperative games is vast; see the relevant chapters in Myerson (1991), Osborne and Rubinstein (1994), or the entire books by Moulin (1988) and Peleg and Sudholter (2003) for textbook presentations.

Let us begin this minicourse by introducing a few central characters in our story. We will be more precise later, but it will be good to start familiarizing ourselves with these terms. Some of this material is borrowed from Serrano (2009):

Game theory: discipline that studies strategic situations.

Cooperative game: strategic situation involving coalitions, whose formation assumes the existence of binding agreements among players.

Characteristic or coalitional function: the most usual way to represent a cooperative game. It prescribes a set of feasible payoffs to each coalition of players.

Solution concept: mapping that assigns predictions to each game.

Nash solution: a solution to bargaining problems that maximizes the product of utility gains from disagreement payoffs.

Kalai-Smorodinsky solution: a solution to bargaining problems that picks the efficient point of the line segment connecting the disagreement point and the utopia point.

Core: solution concept that assigns the set of payoffs that cannot be improved upon by any coalition.

Shapley value: solution concept that assigns the average of marginal contributions to coalitions.

Nucleolus: solution concept that maximizes the welfare of the worst treated coalitions.

These lectures will provide a brief presentation of all these concepts, including illustrations in economic and political applications, and will end with the discussion of some recent advanced topics, focusing on games with coalitional externalities and games with incomplete information.

1.1 Introduction

As you know, we are holding this workshop to celebrate Professor Marilda Sotomayor's 70-th birthday. And it looks like 1944 was a good year for game theory. Indeed, although there were some earlier contributions, the official date of birth of game theory is usually taken to be also 1944, year of publication of the first edition of the monumental *Theory of Games and Economic Behavior* (von Neumann and Morgenstern (1944)). This treatise set the basis for much of the theory of decision making under uncertainty and its multi-person extension to interactive settings.

Game theory is the study of games, also called strategic situations. These are decision problems with multiple decision makers, whose decisions impact one another. It is divided into two branches: noncooperative game theory and cooperative game theory. The actors in noncooperative game theory are individual players, who may reach agreements only if they are self-enforcing. The noncooperative approach provides a rich language and develops useful tools to analyze games. One clear advantage of the approach is that it is able to model how specific details of the interaction among individual players may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt, and it is the subject of this minicourse.

The actors in cooperative game theory are coalitions, that is, groups of players, and in this sense, one can view cooperative game theory as being rooted in its noncooperative counterpart. For the most part, two facts, that coalitions can form and that each coalition has a feasible set of payoffs available to its members, are taken as given. Given the coalitions and their sets of feasible payoffs as primitives, the question tackled is the identification of final payoffs awarded to each player. That is, given a collection of feasible sets of payoffs, one for each coalition, can one predict or recommend a payoff (or set of payoffs) to be awarded to each player? Such predictions or recommendations are embodied in different solution concepts.

One can take several approaches to answering the question just posed. From a positive or descriptive point of view, one may want to get a prediction of the likely outcome of the interaction among the players, and hence, the resulting payoff be understood as the natural consequence of the forces at work in the system. Alternatively, one can take a normative or prescriptive approach, set up a number of normative goals, typically embodied in axioms, and try to derive their logical implications. Authors sometimes disagree on the classification of the different solution concepts according to these two criteria. As we shall see, the understanding of each solution concept is enhanced if one can view it from very distinct approaches. In fact, an important research agenda initiated by John Nash in Nash (1953), referred to as the Nash program, is concerned with developing such connections by exploring noncooperative foundations of the cooperative theory and its solutions. We shall also spend some time discussing results in the Nash program; see Serrano (2005, 2008) for two complementary surveys on the subject.

1.2 Representations of Games: Toward the Characteristic Function

It is useful to begin by presenting the different ways to describe a game. The first two are the usual ways employed in noncooperative game theory.

The most informative way to describe a game is called its *extensive form*. It consists of a game tree, specifying the timing of moves for each player and the information available to each of them at the time of making a move. At the end of each path of moves, a final outcome is reached and a payoff vector is specified. For each player, one can define a *strategy*, i.e., a complete contingent plan of action to play the game. That is, a strategy is a function that specifies a feasible move each time a player is called upon to make a move in the game.

One can abstract from details of the interaction (such as timing of moves and information available at each move), and focus on the concept of strategies. That is, one can list down the set of strategies available to each player, and arrive at the *strategic* or *normal form* of the game. For two players, for example, the normal form is represented in a bimatrix table. One player controls the rows, and the other the columns. Each cell of the bimatrix is occupied with an ordered pair, specifying the payoff to each player if each of them chooses the strategy corresponding to that cell.

One can further abstract from the notion of strategies, which will lead to the *characteristic function form* of representing a game. From the strategic form, one makes assumptions about the strategies used by the complement of a coalition of players to determine the feasible payoffs for the coalition (one can assume that the complement coalition will find ways to minimize the payoffs for players in the coalition, for example). This is the representation most often used in cooperative game theory. There are more general representations, such as the partition function form, which we shall discuss in our last lecture.

Thus, here are the primitives of the basic model in cooperative game theory. Let $N = \{1, \ldots, n\}$ be a finite set of players. Each nonempty subset of N is called a *coalition*. The set N is referred to as the *grand coalition*. For each coalition S, we shall specify a set $V(S) \subset \mathbb{R}^{|S|}$ containing $|S|$-dimensional payoff vectors that are feasible for coalition S. This is called the characteristic function, and the pair (N, V) is called a *cooperative game*. Note how a reduced form approach is taken because one does not explain what strategic choices are behind each of the payoff vectors in $V(S)$. In addition, in this formulation, it is implicitly assumed that the actions taken by the complement coalition (those players in $N \setminus S$) cannot prevent S from achieving each of the payoff vectors in $V(S)$.

With suitable regularity assumptions one can make, $\partial V(S)$ is the Pareto frontier of

$V(S)$, i.e., the set of vectors $x_S \in V(S)$ such that there does not exist $y_S \in V(S)$ satisfying that $y_i \geq x_i$ for all $i \in S$ with at least one strict inequality.

Other assumptions usually made relate the possibilities available to different coalitions. Among them, a very important one is *balancedness*, which we shall define next:

A collection $\mathcal{T}$ of coalitions is balanced if there exists a set of weights $w(S) \in [0,1]$ for each $S \in \mathcal{T}$ such that for every $i \in N$, $\sum_{S \in \mathcal{T}, S \supseteq \{i\}} w(S) = 1$. One can think of these weights as the fraction of time that each player devotes to each coalition he is a member of, with a given coalition representing the same fraction of time for each player. The game (N, V) is balanced if $x_N \in V(N)$ whenever $(x_S) \in V(S)$ for every S in a balanced collection $\mathcal{T}$. That is, the grand coalition can always implement any "time-sharing arrangement" that the different subcoalitions may come up with.

The characteristic function defined so far is often referred to as a *nontransferable utility (NTU)* game. A particular case is the *transferable utility (TU)* game case, in which for each coalition $S \subseteq N$, there exists a real number $v(S)$ such that

$$V(S) = \{x \in I\!R^{|S|} : \sum_{i \in S} x_i \leq v(S)\}.$$

Abusing notation slightly, we shall denote a TU game by (N, v). In the TU case there is an underlying nummeraire –money– that can transfer utility or payoff at a one-to-one rate from one player to any other.

Another particular case of NTU games is the class of *pure bargaining problems*, in which intermediate coalitions between individuals and the Grand Coalition have no power. That is, for every S containing at least two players, $S \neq N$, $V(S) \subseteq \{V(\{i\})\}_{i \in S} - I\!R_+^{|S|}$.

1.3 Solution Concepts

Given a characteristic function, i.e., a collection of sets $V(S)$, one for each S, the theory formulates its predictions on the basis of different *solution concepts.* We shall concentrate on the case in which the grand coalition forms, that is, cooperation is totally successful.

Of course, solution concepts can be adapted to take care of the case in which this does not happen.

A *solution* is a mapping that assigns a set of payoff vectors in $V(N)$ to each characteristic function game (N, V). Thus, a solution in general prescribes a set, which can be empty, or a singleton (when it assigns a unique payoff vector as a function of the fundamentals of the problem). The leading set-valued cooperative solution concept is the core, while some of the most used single-valued ones are the Shapley value for TU games or the Nash solution for pure bargaining problems.

1.4 Bargaining

Nash (1950) is the pioneering work in the axiomatic theory of bargaining. He was not very clear as to whether it should have either a positive or a normative interpretation, although today most authors give it a normative content. The starting point is to list down a series of properties (axioms) that a solution to a bargaining problem should have. The question asked is whether one can isolate a solution or a class of solutions that satisfies these properties.

Although we could employ the general notation of NTU games, we present here the usual notation for bargaining problems. A two-person bargaining problem is a pair (U, d), where $U \subseteq \mathbb{R}^2_+$ is the feasible set of utility pairs available to the bargainers, and $d \in \mathbb{R}^2_+$ is the disagreement point. We shall assume that U is compact, convex, comprehensive, and that it contains a point $x \gg d$. Denote by Σ^2 the class of all such two-person bargaining problems.

A bargaining solution is a function $f(U) : \Sigma^2 \to U$. Note that nonemptiness and uniqueness are built into the definition.

Nash proposed the following list of axioms that a solution should satisfy:

A1. Scale invariance: Let $\alpha \in \mathbb{R}^2_{++}$ and $\beta \in \mathbb{R}^2$. Denote by U' the set of utility pairs (u'_1, u'_2) satisfying that there exists $(u_1, u_2) \in U$ such that $u'_1 = \alpha_1 u_1 + \beta_1$ and

$u_2' = \alpha_2 u_2 + \beta_2$. Also, let $d' = (d_1', d_2')$ be such that $d_1' = \alpha_1 d_1 + \beta_1$ and $d_2' = \alpha_2 d_2 + \beta_2$. Then, scale invariance says that $f(U', d') = (\alpha_1 f_1(U, d) + \beta_1, \alpha_2 f_2(U, d) + \beta_2)$. Thanks to scale invariance, we can always take $\beta = -d$ and normalize the disagreement point to 0.

A2. Symmetry: If U is a symmetric problem with respect to the 45 degree line and $d_1 = d_2$, then $f_1(U, d) = f_2(U, d)$.

A3. Pareto efficiency: There does not exist $y \in U$ such that $y \gg f(U, d)$.

A4. Independence of irrelevant alternatives (IIA): Let $T \subseteq U$ and $f(U, d) \in T$. Then, $f(T, d) = f(U, d)$.

Theorem (Nash, 1950): There exists a unique bargaining solution that satisfies A1-A4. It is $n(U, d) = \arg\max_{x \in U} (x_1 - d_1)(x_2 - d_2)$.

Proof: It is easy to check that n satisfies A1-A4. We prove the other part. Let f be a solution that satisfies A1-A4. We need to prove that $f = n$.

Consider an arbitrary bargaining problem (U, d). Since f satisfies A1, we can normalize the disagreement point to 0, and speak of U as the bargaining problem. Compute $n(U)$ and transform the set U into T by dividing utilities by $n(U)$. That is, each point $(u_1, u_2) \in U$ is transformed into $(u_1/n_1(U), u_2/n_2(U))$. Clearly, $n(T) = (1, 1)$.

Consider now the bargaining problem R, whose Pareto frontier is given by the supporting line to the hyperbola $u_1 u_2 = 1$ at the point $(1, 1)$. By A2 and A3, $f(R) = (1, 1)$. By A4, $f(T) = (1, 1) = n(T)$. Finally, by A1, $f(U) = n(U)$. Q.e.d.

(fig. c1) Draw a set S (note the picture is drawn denoting the bargaining problem by S instead of U), nonsymmetric, and draw n by drawing the highest hyperbola tangent to the frontier of the set. Draw below the set T, where we have divided S by $n(S)$. Draw $n(T) = (1, 1)$ and the hyperbola tangent there. Draw the set R: the right angle triangle that supports (1,1). The aspiration levels are also labelled.

A4 has been the target of much criticism. Much of it can be summarized in the following example:

(fig. c2) Draw a right angle triangle S (same comment as in fig. C1) and the hyperbola tangent at $(1, 1)$. Draw $T \subseteq S$ as the set of points where 2's utility is at most 1. Label $n(S) = n(T) = (1, 1)$. Clearly, the Nash solution fails to take bargainers' aspiration levels into account when determining what the outcome should be.

Kalai and Smorodinsky (1975) introduce the following axiom, an alternative to IIA:

A5. Individual monotonicity: Let $a_i(U)$ be bargainer i's aspiration level, defined by $a_i(U) = \max x_i$ such that $(x_i, d_j) \in U$. Suppose U and T are two bargaining problems such that $a_j(U) = a_j(T)$ and $a_i(U) \geq a_i(T)$. Then, $f_i(U) \geq f_i(T)$.

Denote by $\partial(U)$ the Pareto frontier of U, and by $[x, y]$ the straight line segment connecting x and y.

Theorem (Kalai and Smorodinsky, 1975): There exists a unique solution that satisfies A1-A3 and A5. It is $ks(U) = \partial(U) \cap [a(U), d]$.

Proof: It is easy to check that ks satisfies A1-A3 and A5. We prove the other direction. Let f be a solution that satisfies A1-A3 and A5, and consider an arbitrary problem U. (By A1, we can again normalize $d = 0$). Transform U into T by dividing agent 1's utility scale by $a_1(U)$ and 2's by $a_2(U)$. By A1, $f(T) = f(U)/a(U)$. Also, $a_1(T) = a_2(T) = 1$. Calculate $ks(T)$. Note that $ks_1(T) = ks_2(T)$.

Now construct the set R, whose Pareto frontier is $[(1, 0), ks(T)] \cup [(0, 1), ks(T)]$. Since R is symmetric, by A2 and A3, $f(R) = ks(R) = ks(T)$. But then, by A5, $f(T) = ks(T)$, and then, by A1 of f and of ks, $f(U) = ks(U)$. Q.e.d.

(fig. c3) Draw first an arbitrary set S (same comment as in fig. C1) and label the aspiration levels $a_1(S)$ and $a_2(S)$. Below, draw the transformed set T, where we have divided S by $a(S)$. Draw the vertical through (1,0) and the horizontal through (0,1) and connect their intersection with the origin. Label $ks(T)$. Draw also the set R, whose Pareto frontier connects (1,0) and (0,1) to $ks(T)$.

These characterizations have assumed a fixed number of bargainers. An interesting property that has proved quite powerful requires to talk about problems with a variable number of agents. This is the property referred to as consistency.

Let Σ^n be the class of all n-person bargaining problems with the properties described above, and let $\Sigma = \cup_{n \in I\!N} \Sigma^n$.

A solution $f(U)$ is now a function $f(U) : \Sigma \to U$. Let N be a set of n agents. Given a vector $x \in I\!R^n$ and $P \subseteq N$, let $x_P = (x_i)_{i \in P}$. Let $U \in \Sigma^n$ and let $[U|x_{N \setminus P}] \in \Sigma^p$ denote the projection of U onto $I\!R^p$ when we fix the other $n - p$ components at $x_{N \setminus P}$.

We shall introduce the following two axioms:

A6. Anonymity: Let $\pi : N \to N$ be a bijection. Then, $f(\pi U, \pi d) = \pi[f(U, d)]$. This is a slight strenghthening of symmetry, since it applies also to nonsymmetric problems. It says that names do not matter.

A7. Consistency: For any $n \in I\!N$, let $x = f(U)$, where $U \in \Sigma^n$. Then, for every $P \subseteq N$, $f(U|x_{N \setminus P}) = x_P$. That is, the solution should be invariant to the number of players in the problem.

Theorem (Lensberg, 1988): There exists a unique solution that satisfies A1, A3, A6 and A7. It is the Nash solution.

The proof is rather involved, but here is an intuition for it. First, it is easy to check that the Nash solution satisfies the axioms. The difficult part of the proof relies on two lemmata. Define Σ^2_{el} the class of all two-person problems U such that $n_1(U) = n_2(U)$ satisfying that the Pareto frontier contains a nontrivial line segment around the Nash solution.

Lemma 1: Consider the class Σ^2_{el}. If f satisfies A1, A3, A6 and A7, $f = n$ on this class.

A sketch of the proof is as follows: By A1, we can normalize the Nash solution to be (1,1). Then, we construct a set in a higher dimension, where we raise our set by a height of 1 (the shorter the straight line segment around (1,1) is, the higher the dimension we will need to go; intuitively, more courvature has to be taken care of when we slice back).

We then take the permutation of such an enlarged set, and look at the union of all such permuted sets. This set is such that all its permutations leave unchanged a nontrivial section of the symmetric hyperplane around $(1, \ldots, 1)$. By A3 and A6, the solution f must assign the vector of all ones. When we slice the set at that point, we get the original set (here it was crucial to be able to go up an arbitrary number of dimensions). By A7, $f = n$ over Σ^2_{el}.

Lemma 2: If f satisfies A1, A3, A7 and $f(U) = n(U)$ whenever $U \in \Sigma^2_{el}$, then $f(U) = n(U)$ for every $U \in \Sigma$.

The proof relies upon the fact that any set U can be obtained as the slice of a set in higher dimensions formed by the hyperplane supporting $(1, 1, \ldots, 1)$ (after raising U by one unit) intersected with the negative orthant of $(1 + \epsilon, \ldots, 1 + \epsilon)$ for some ϵ. The trick is then to construct such a set. When we slice back to two dimensions, we get a set in Σ^2_{el}, where $f = n$. This is true for every pair, and yields the result.

A plausible positive theory of bargaining is based on the alternating offers extensive forms studied in Rubinstein (1982), Binmore, Rubinstein and Wolinsky (1986) and Krishna and Serrano (1996). There, it is shown that the unique SPE outcome of the game with discount factor δ is described by the equations

$$u_i(x_i) = \delta u_i(x_i + \alpha) \quad \forall i \in N$$

and

$$\sum_{i \in N} x_i + \alpha = q,$$

where q is the size of the pie to be split. Clearly, as $\delta \to 1$, this unique outcome converges to the Nash solution.

Theorem (Krishna and Serrano, 1996): The Unique SPE agreement of the noncooperative game of alternating offers with exit converges to the Nash solution as $\delta \to 1$.

Proof: Let $((x_i(\delta))_{i \in N}, \alpha(\delta))$ be the unique SPE outcome of the game of discount factor δ. Choose player i and player j. From the equations, it is true that:

$$u_i(x_i(\delta) + \alpha(\delta))/u_i(x_i(\delta)) = u_j(x_j(\delta) + \alpha(\delta))/u_j(x_j(\delta)) = \delta,$$

or

$$u_i(x_i(\delta)+\alpha(\delta)) \cdot u_j(x_j(\delta)) \cdot \prod_{k\neq\{i,j\}} u_k(x_k(\delta)) = u_i(x_i(\delta)) \cdot u_j(x_j(\delta)+\alpha(\delta)) \cdot \prod_{k\neq\{i,j\}} u_k(x_k(\delta)).$$

Further, note that for all δ, the SPE agreement is efficient, and that, as $\delta \to 1$, $\alpha(\delta) \to 0$. Thus, $\lim_{\delta\to 1}(x_i(\delta))_{i\in N}$ is the Nash solution. Q.e.d.

2 Lecture 2: The Core

In today's lecture about the core, we mostly restrict attention to the class of TU games, although some of the results we will obtain extend beyond it.

The set of allocations of the game (N, v) (adopting the usual abuse of notation) is the set of payoffs

$$X(N,v) = \{x \in I\!R^n : \sum_{i\in N} x_i = x(N) \leq v(N)\}.$$

The set of imputations of the game (N, v) is the set of its individually rational allocations:

$$X_0(N,v) = \{x \in X(N,v) : x_i \geq v(\{i\}) \forall i \in N\}.$$

The core of the game (N, v) is the following set of allocations:

$$C(N,v) = \{x \in X(N,v) : \forall S \subseteq N, x(S) \geq v(S)\}.$$

That is, it is the set of payoffs that cannot be improved upon by any coalition of players.

For some games, the core may be empty. For example, consider a 3- person majority game, where for every two- and three-person coalitions S, $v(S) = 1$, and $v(\{i\}) = 0 \forall i \in N$. One important question we can ask is whether there are some classes of TU games where the core is always nonempty. We will tackle this question now, recalling the assumption of balancedness.

Theorem (Bondareva, 1963 - Shapley, 1967): $C(N, v)$ is nonempty if and only if (N, v) is balanced.

The proof relies heavily on the duality theorem of linear programming: a primal program has a solution if and only if so does its dual. Furthermore, the optimal value in the two programs must coincide. It turns out that one can write the nonemptiness of the core conditions as a solution to a certain dual program, whose primal will have a solution whenever the balancedness condition is met.

For the general class of NTU games, balancedness is only sufficient for nonemptiness of the core:

Theorem (Scarf, 1967): If the game (N, V) is balanced, the core is nonempty.

Now we turn to an axiomatic characterization of the core. We should emphasize that the origins of the core –Edgeworth (1881)– (unlike those of the Nash solution) were not axiomatic. The core is an appealing solution concept by its very definition. However, once this is accepted, it is of interest to ask what axioms lie behind this appealing solution concept. To the extent we like the solution, so should we its underlying principles. A list of the axioms needed follows.

Consider a class of TU games Γ. Let $f : \Gamma \to X(N, v)$ be a solution, which specifies for each game (N, v) a nonempty set $f(N, v) \subseteq X(N, v)$.

C1. Individual rationality: For all $(N, v) \in \Gamma$, $f(N, v) \subseteq X_0(N, v)$.

C2. Superadditivity: Let (N, v_1) and $(N, v_2) \in \Gamma$. Denote the sum game by u such that for all $S \subseteq N$, $u(S) = v_1(S) + v_2(S)$. Then, $f(N, v_1) + f(N, v_2) \subseteq f(N, u)$. In some sense, the solution excludes negative externalities between two scenarios when we combine them.

C3. DM consistency: For every $(N, v) \in \Gamma$ and every $x \in f(N, v)$, we have that for all $S \subseteq N$, $x_S \in f(S, v_{x,S})$. The Davis-Maschler reduced game is defined as follows: For every $S \subseteq N$ and $x \in \mathbb{R}^n$, the game $(S, v_{x,S})$ is such that $v_{x,S}(S) = v(N) - x(N \backslash S)$ and for every nonempty $T \subseteq S$, $T \neq S$, $v_{x,S}(T) = \max_{Q \subseteq N \backslash S} \{v(T \cup Q) - x(Q)\}$. This is the good old property of consistency, provided that those who stay in the reduced game can hire the resources of those who left by paying them at the rate specified by the allocation they accepted.

Theorem (Peleg, 1986): Consider the class Γ of TU games. Then, a solution f on Γ satisfies C1-C3 if and only if $f(N, v) = C(N, v)$ for every $(N, v) \in \Gamma$.

Proof: As usual, we will not spend time checking that the core satisfies the three axioms. We shall prove the other part, i.e., let f satisfy C1-C3 and we shall prove that $f(N, v) = C(N, v)$ for every game $(N, v) \in \Gamma$.

Lemma 1: If f satisfies C1 and C3, f is efficient.

Proof of Lemma 1: Let $x \in f(N, v)$. By C3, $x_i \in f(\{i\}, v_{x,\{i\}})$. Recall that $v_{x,\{i\}}(\{i\}) = v(N) - x(N \backslash \{i\})$. By C1, $x_i \geq v_{x,\{i\}} = v(N) - x(N \backslash \{i\})$. By definition of f, $x_i \leq v_{x,\{i\}}$. Thus, we have equality: $x_i = v(N) - x(N \backslash \{i\})$.

Remark: Thus we know that for every game with $|N| = 1$, if f satisfies C1 and C3, $f(N, v) = C(N, v)$. And if $|N| = 2$, $f(N, v) \subseteq C(N, v)$.

Lemma 2: If f satisfies C1 and C3, $f(N, v) \subseteq C(N, v)$ for every $(N, v) \in \Gamma$.

Proof of Lemma 2: By induction on the number of players. By the previous remark, the Lemma holds for one-player and two-player games. Consider then a game (N, v) with $|N| \geq 3$. We will need the following property of the core:

Property: The core satisfies converse consistency. Consider the game (N, v) and let $x(N) = v(N)$. Suppose that for all $S \subseteq N, |S| = 2$, $x_S \in C(S, v_{x,S})$. Then, $x \in C(N, v)$. (Prove this as an exercise).

Then take $x \in f(N, v)$. By Lemma 1, $x(N) = v(N)$. By C3, for all $S \subseteq N, |S| = 2$, $x_S \in f(S, v_{x,S})$. By the remark before this lemma, for all $S \subseteq N, |S| = 2$, $x_S \in C(S, v_{x,s})$. Finally, by the property just stated, $x \in C(N, v)$, as we wanted to show.

Lemma 3: If f satisfies C1-C3, $C(N, v) \subseteq f(N, v)$ for every $(N, v) \in \Gamma$.

Proof of Lemma 3: Consider the game (N, v) and $x \in C(N, v)$. We know this is possible since we have already proved that $f(N, v) \subseteq C(N, v)$ and Γ is a class where $f(N, v) \neq \emptyset$.

Construct the game (N, w) as follows: $w(\{i\}) = v(\{i\}) \forall i \in N$ and $w(S) = x(S)$ for all other S. Note that $C(N,w) = \{x\}$. Now construct the game (N,u), where $u = v - w$, i.e., $u(\{i\}) = 0 \forall i \in N$ and $u(S) = v(S) - x(S)$ for all other S. Note that $C(N,u) = \{0\}$.

By Lemma 2, $f(N,w) = \{x\}$ and $f(N,u) = \{0\}$. By C2, $f(N,v)$ contains $f(N,w) + f(N,u)$. Thus, $x \in f(N,v)$. Q.e.d.

Peleg (1985) provides a related axiomatization of the core for the class of NTU games.

Now we will spend time talking about an important class of TU games, that of convex games. Intuitively, convexity means increasing returns to cooperation.

A game (N,v) is convex if for every S, T, i, $S \subseteq T, i \notin T$, we have that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

Let π be a permutation of the players. Define the vector of marginal contributions x^π given π as follows: (suppose $\pi = (1, 2, \ldots, n)$ to simplify notation). Then, $x_1^\pi = v(\{1\})$, $x_2^\pi = v(1,2) - v(1)$, $x_3^\pi = v(1,2,3) - v(1,2)$, ..., $x_n^\pi = v(N) - v(N \backslash \{n\})$.

Theorem (Shapley, 1971): Let (N,v) be convex. Then, $C(N,v)$ is the convex hull of the vectors x^π for every possible order π.

Proof: We will prove one direction. The other is considerably harder. We will prove that the convex hull of the vectors x^π is contained in the core.

Fix π, say, $\pi = (1, 2, \ldots, n)$. We will prove that $x^\pi \in C(N,v)$ if (N,v) is convex. Of course, $x^\pi(N) = v(N)$. Take now $S \subseteq N, S \neq N$, $S = \{i_1, i_2, \ldots, i_s\}$ arranged so that $i_1 < i_2 < \ldots < i_s$. By convexity, we know that:

$$v(S) - v(S \backslash \{i_s\}) \leq v(1, 2, \ldots, i_s) - v(1, 2, \ldots, i_s - 1) = x_{i_s}^\pi.$$

$$v(S \backslash \{i_s\}) - v(S \backslash \{i_s, i_{s-1}) \leq v(1, 2, \ldots, i_{s-1}) - v(1, 2, \ldots, i_{s-1} - 1) = x_{i_{s-1}}^\pi,$$

...

$$v(i_1, i_2) - v(i_1) \leq v(1, 2, \ldots, i_2) - v(1, 2, \ldots, i_2 - 1) = x_{i_2}^\pi,$$

$$v(i_1) - v(\emptyset) \leq v(1, 2, \ldots, i_1) - v(1, 2, \ldots, i_1 - 1) = x_{i_1}^\pi.$$

Adding up all these inequalities, we get that $v(S) \leq x^\pi(S)$. Since S was arbitrary, $x^\pi \in C(N,v)$. Q.e.d.

Let us turn now to a noncooperative implementation of the core of convex games. We interpret the problem underlying the characteristic function as follows: each player i holds a productive asset. We interpret $v(S)$ as the utility value for any player of holding the portfolio of assets S. In addition, each agent has an amount of money (normalized to 0, so we allow it to be negative).

Let $q \in \Delta^n$, where Δ^n is the $(n-1)$-dimensional probability simplex. Consider the game G_q:

Stage 0. Nature moves and appoints player i to be the broker with probability q_i, $i = 1, \ldots, n$.

Stage 1. In the subgame where player i is the broker, player i proposes x such that $x(N) = v(N)$.

Stage 2. Following a specified protocol P, players in $N \setminus \{i\}$ respond sequentially to x. Player j's messages are either Y or [N,B_j].

Trade stage, not part of the game . If player j says Y, he sells his asset to player i at the price x_j. If j says [N,B_j], he will buy from player i the largest subset of the assets B_j that i has. All trade with the broker happens following P.

Denote by E_q the set of SPE payoffs of the game G_q.

Theorem (Serrano, 1995b): Let (N, v) be strictly convex. Then, $C(N, v) = \cup_{q \in \Delta^n} E_q$.

Proof: Let $M_i \subseteq C(N, v)$ be the set of player i's payoff maximizers within the core. By Shapley's theorem, if $x \in M_i$, $x_i = v(N) - v(N \setminus \{i\})$.

Step 1: $\cup_{q \in \Delta^n} E_q \subseteq C(N, v)$.

Proof of step 1: Let $y \in \cup_{q \in \Delta^n} E_q$. Thus, there exist q such that $y \in E_q$. Consider then the game G_q. The following two claims can then be established:

Claim 1: In the subgame of G_q where i is the broker, let i propose x. Then, x is unanimously accepted if and only if $x \in C(N, v)$.

Claim 2: In the subgame of G_q where i is the broker, if the proposal $x \notin C(N,v)$, player i's payoff in this subgame is strictly less than $v(N) - v(N \backslash \{i\})$.

These two claims establish that $z(i)$ is a SPE payoff of the subgame where i is the broker if and only if $z(i) \in M_i$. Moreover, $y = \sum_{i \in N} q_i z(i)$. Since $C(N,v)$ is a convex set, $y \in C(N,v)$.

Step 2: $C(N,v) \subseteq \cup_{q \in \Delta^n} E_q$.

Proof of step 2: Let $y \in C(N,v)$. By Shapley's theorem, there exist nonnegative weights α_π adding up to 1 such that $y = \sum_\pi \alpha_\pi x^\pi$. Note that for every π, $x^\pi \in M_i$, where $\pi_n = i$. Then for each i, choose q_i to be the sum of α_π over all those π where $\pi(N) = i$. Clearly, $y \in E_q$. Q.e.d.

Next, we turn to the connections with general equilibrium theory. In economics, the institution of markets and the notion of prices are essential to the understanding of the allocation of goods and the distribution of wealth among individuals. For simplicity in the presentation, we shall concentrate on exchange economies, and disregard production aspects. That is, we shall assume that the goods in question have already been produced in some fixed amounts, and now they are to be allocated to individuals to satisfy their consumption needs.

An *exchange economy* is a system in which each agent i in the set N has a consumption set $Z_i \subseteq \mathbb{R}^l_+$ of commodity bundles, as well as a preference relation over Z_i and an initial endowment $\omega_i \in Z_i$ of the commodities. A feasible *allocation* of goods in the economy is a list of bundles $(z_i)_{i \in N}$ such that $z_i \in Z_i$ and $\sum_{i \in N} z_i \leq \sum_{i \in N} \omega_i$. An allocation is *competitive* if it is supported by a *competitive equilibrium*. A competitive equilibrium is a price-allocation pair $(p, (z_i)_{i \in N})$, where $p \in \mathbb{R}^l \setminus \{0\}$ is such that

- for every $i \in N$, z_i is top-ranked for agent i among all bundles z satisfying that $pz \leq p\omega_i$,
- and $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$.

In words, this is what the concept expresses. First, at the equilibrium prices, each agent demands z_i, i.e., wishes to purchase this bundle among the set of affordable bundles, the budget set. And second, these demands are such that all markets clear, i.e., total demand equals total supply.

Note how the notion of a competitive equilibrium relies on the principle of private ownership (each individual owns his or her endowment, which allows him or her to access markets and purchase things). Moreover, each agent is a price-taker in all markets. That is, no single individual can affect the market prices with his or her actions; prices are fixed parameters in each individual's consumption decision. The usual justification for the price-taking assumption is that each individual is "very small" with respect to the size of the economy, and hence, has no market power.

One difficulty with the competitive equilibrium concept is that it does not explain where prices come from. There is no single agent in the model responsible for coming up with them. Walras (1874) told the story of an auctioneer calling out prices until demand and supply coincide, but in many real-world markets there is no auctioneer. More generally, economists attribute the equilibrium prices to the workings of the forces of demand and supply, but this appears to be simply repeating the definition. So, is there a different way one can explain competitive equilibrium prices?

As it turns out, there is a very robust result that answers this question. We refer to it as the *equivalence principle* (see, e.g., Aumann (1987)), by which, under certain regularity conditions, the predictions provided by different game-theoretic solution concepts, when applied to an economy with a large enough set of agents, tend to converge to the set of competitive equilibrium allocations. One of the first results in this tradition was provided by Edgeworth in 1881 for the core. Note how the core of the economy can be defined in the space of allocations, using the same definition as above. Namely, a feasible allocation is in the core if it cannot be blocked by any coalition of agents when making use of the coalition's endowments.

Edgeworth's result was generalized later by Debreu and Scarf (1963) for the case in which an exchange economy is replicated an arbitrary number of times (Anderson (1978) studies the more general case of arbitrary sequences of economies, not necessarily replicas). An informal statement of the Debreu-Scarf theorem follows:

Theorem (Debreu and Scarf, 1963): Consider an exchange economy. Then,

(i) The set of competitive equilibrium allocations is contained in the core.

(ii) For each noncompetitive core allocation of the original economy, there exists a sufficiently large replica of the economy for which the replica of the allocation is blocked.

The first part states a very appealing property of competitive allocations, i.e., their coalitional stability. The second part, known as the core convergence theorem, states that the core "shrinks" to the set of competitive allocations as the economy grows large.

Aumann (1964) models the economy as an atomless measure space, and demonstrates the following core equivalence theorem:

Theorem (Aumann, 1964): Let the economy consists of an atomless continuum of agents. Then, the core coincides with the set of competitive allocations.

We shall close this lecture with a political application. Consider majority games within a parliament. Suppose there are 100 seats, and decisions are made by simple majority so that 51 votes are required to pass a piece of legislation.

In the first specification, suppose there is a very large party –player 1–, who has 90 seats. There are five small parties, with 2 seats each. Given the simple majority rules, this problem can be represented by the following TU characteristic function: $v(S) = 1$ if S contains player 1, and $v(S) = 0$ otherwise. The interpretation is that each winning coalition can get the entire surplus –pass the desired proposal. Here, a coalition is winning if and only if player 1 is in it. For this problem, the core is a singleton: the entire unit of surplus is allocated to player 1, who has all the power. Any split of the unit surplus of the grand coalition ($v(N) = 1$) that gives some positive fraction of surplus to any of the small parties can be blocked by the coalition of player 1 alone.

Consider now a second problem, in which player 1, who continues to be the large party, has 35 seats, and each of the other five parties has 13 seats. Now, the characteristic function is as follows: $v(S) = 1$ if and only if S either contains player 1 and two small parties, or it contains four of the small parties; $v(S) = 0$ otherwise. It is easy to see that now the core is empty: any split of the unit surplus will be blocked by at least one coalition. For example, the entire unit going to player 1 is blocked by the coalition of all five small parties, which can award 0.2 to each of them. But this arrangement, in which each small party gets 0.2 and player 1 nothing, is blocked as well, because player 1 can bribe two of the small parties (say, players 2 and 3) and promise them 1/3 each, keeping the other third for itself, and so on. The emptiness of the core is a way to describe the fragility of any agreement, due to the inherent instability of this coalition formation game.

3 Lecture 3: Values for TU Games

In this third lecture, we shall concentrate on single-valued solutions for TU games, which are sometimes referred to as values. The two most important values will be covered, i.e., the Shapley value and the nucleolus.

3.1 The Shapley Value

Now consider a transferable utility or TU game in characteristic function form. The number $v(S)$ is referred to as the worth of S, and it expresses S's initial position (e.g., the maximum total amount of surplus in nummeraire –money, or power– that S initially has at its disposal.

Shapley (1953) is interested in solving in a fair and unique way the problem of distribution of surplus among the players, when taking into account the worth of each coalition. To do this, he restricts attention to single-valued solutions and resorts to the axiomatic method.

Consider the class Γ of all TU games. A value f (or single-valued solution) is a function $f : \Gamma \to I\!R^n$. We shall impose the following properties on f:

S1. Efficiency: For all $(N, v) \in \Gamma$, $\sum_{i \in N} f_i(N, v) = v(N)$.

S2. Anonymity: Let $\pi : N \to N$ be a bijection. Then, $f(\pi N, \pi v) = \pi[f(N, v)]$.

S3. Additivity: Consider the games (N, v_1) and (N, v_2). Let the sum game (N, v) be defined as follows: for every $S \subseteq N$, $v(S) = v_1(S) + v_2(S)$. Then, $f(N, v) = f(N, v_1) + f(N, v_2)$.

S4. Null or dummy player: Let $i$be such that $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subseteq N$. Then, $f_i(N, v) = v(\{i\})$. That is, if someone contributes zero to all coalitions, he should be paid 0.

Theorem (Shapley, 1953): There exists a unique value f satisfying S1-S4. It is the Shapley value $Sh(N, v)$, where

$$Sh_i(N, v) = \sum_{S \subseteq N} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \backslash \{i\})].$$

Before we prove the theorem, let us see an example. Consider the 3- player game, where $v(N) = 3$, $v(1, 2) = 3$, $v(1, 3) = 2$, $v(2, 3) = 1$ and $v(i) = 0 \forall i \in N$. Then, $Sh(N, v) = (1.5, 1, 0.5)$.

Proof: First, as always, you can check that the above formula satisfies S1-S4. We prove the other part. To do this, we first introduce the class of simple games.

A game (N, v_R) is simple if there exists a unique $R \subseteq N, R \neq \emptyset$ such that $v_R(S) = 1$ if $R \subseteq S$, and $v_R(S) = 0$ otherwise. That is, each player in R has veto power. If $|R| = 1$, the game is dictatorial. Note that there are exactly $2^{|N|} - 1$ simple games in the class of n-player games. Now we should think of a game (N, v) as an element of $I\!R^{2^{|N|}-1}$ that assigns to each coalition $S \in 2^N$ a real number. In fact, the class of all n-player games is nothing but this Euclidean space: note that the sum and the product by a scalar of a game give us another game, the conditions of a linear space.

Lemma: Let Γ_s be the class of simple games. Γ_s constitutes a basis of $\Gamma = I\!R^{2^{|N|}-1}$ (the class of n- player TU games).

Proof of Lemma: We need to show that the $2^{|N|}-1$ vectors v_R for every $R \subseteq N, R \neq \emptyset$ are linearly independent. That is, if $\sum_{R \subseteq N, R \neq \emptyset} \alpha_R v_R = 0$, then $\alpha_R = 0 \forall R$. Suppose not: that is, suppose that $\sum_{R \subseteq N, R \neq \emptyset} \alpha_R v_R = 0$, but $\alpha_R \neq 0$ for some R'. Denote by T a minimal coalition in this class: T is such that $\alpha_T \neq 0$ and for all $S \subseteq T$, $\alpha_S = 0$. T exists because we have a finite number of coalitions.

Thus, we can write $v_T = -\sum_{R \neq T} \frac{\alpha_R}{\alpha_T} v_R$. Now consider the entry corresponding to coalition T. On the left hand side, $v_T(T) = 1$. The right hand side, however, equals 0: if $R \subseteq T, R \neq T$, $\alpha_R = 0$, and $v_R(T) = 0$ otherwise. This is a contradiction that proves the lemma.

Thus, for every game $v \in \Gamma$, there exist unique scalars β_R such that $v = \sum_{R \subseteq N, R \neq \emptyset} \beta_R v_R$.

To prove the theorem, consider a game v_R for a fixed $R \subseteq N$. If f satisfies S1, S2 and S4, $f(N, v_R)$ awards $1/|R|$ to each member of R, and 0 to the rest. Thus, by S3, $f(N, v)$ is uniquely defined, given the unique coordinates of the game v. Q.e.d.

What is especially surprising in Shapley's result is that nothing in the axioms (with the possible exception of the dummy axiom) hints at the idea of marginal contributions, so marginality in general is the outcome of all the axioms, including additivity or linearity. Among the axioms utilized by Shapley, additivity is the one with a lower normative content: it is simply a mathematical property to justify simplicity in the computation of the solution. Young (1985) provides a beautiful counterpart to Shapley's theorem. He drops additivity (as well as the dummy player axiom), and instead, uses an axiom of marginality. Marginality means that the solution should pay the same to a pleyar in two games if his or her marginal contributions to coalitions is the same in both games. Marginality is an idea with a strong tradition in economic theory. Young's result is "dual" to Shapley's, in the sense that marginality is assumed and additivity derived as the result:

Theorem (Young, 1985): There exists a unique single-valued solution to TU games satisfying efficiency, symmetry and marginality. It is the Shapley value.

Apart from these two, Hart and Mas-Colell (1989) provides further axiomatizations of the Shapley value using the idea of potential and the concept of consistency, as described in the previous section.

There is no single way to extend the Shapley value to the class of NTU games. There are three main extensions that have been proposed: the Shapley λ-transfer value (Shapley (1969)), the Harsanyi value (Harsanyi (1963)), and the Maschler-Owen consistent value (Maschler and Owen (1992)).

As was the case for the core, there is a value equivalence theorem. The result holds for the TU domain (see Aumann and Shapley (1974)). It can be shown that the Shapley value payoffs can be supported by competitive prices. Furthermore, in large enough economies, the set of competitive payoffs "shrinks" to approximate the Shapley value. However, the result cannot be easily extended to the NTU domain.

What about the noncooperative implementation of the Shapley value? Consider the following bargaining game, proposed in Hart and Mas- Colell (1996).

The procedure G_ρ: The proposer, chosen at random with equal probability, makes a proposal (a split of $v(N)$). If it is unanimously accepted, it is implemented. Otherwise, the proposer is eliminated with probability $1-\rho$ together with her resources, while with probability ρ the n-player procedure is repeated. If the proposer is eliminated, the same procedure is played among the $n-1$ remaining players.

Theorem (Hart and Mas-Colell, 1996): For any ρ such that $0 \leq \rho < 1$, the set of stationary SPE payoffs of G_ρ is a singleton: $Sh(N, v)$.

The proof is not too complicated, given the stationarity imposed on the strategies.

One way to understand the Hart and Mas-Colell implementation result is to check that the rules of the procedure and stationary behavior in it are in agreement with Shapley's axioms. That is, the equilibrium relies on immediate acceptances of proposals, stationary strategies treat substitute players similarly, the equations describing the equilibrium have an additive structure, and dummy players will have to receive 0 because no resources are destroyed if they are asked to leave. It is also worth stressing the important role in the procedure of players' marginal contributions to coalitions: following a rejection, a proposer incurs the risk of being thrown out and the others of losing his resources, which seem to suggest a "price" for them.

Krishna and Serrano (1995) analyze the same game without imposing stationarity on the strategies, and find that for $\rho < 1/(n-1)$, uniqueness obtains. Also, Pérez-Castrillo and Wettstein (2001) uses a variant of the Hart and Mas-Colell procedure, by replacing the random choice of proposers with a bidding stage, in which players bid to obtain the right to make proposals.

We shall close our brief discussion of the Shapley value with an application.

Consider again the class of majority problems in a parliament consisting of 100 seats. As we shall see, the Shapley value is a good way to understand the power that each party has in the legislature.

Let us begin by considering again the problem in which player 1 has 90 seats, while each of the five small parties has 2 seats. It is easy to see that the Shapley value, like the core in this case, awards the entire unit of surplus to player 1: effectively, each of the small parties is a dummy player, and hence, the Shapley value awards zero to each of them.

Consider a second problem, in which player 1 is a big party with 35 seats, and there are 5 small parties, with 13 seats each. The Shapley value awards 1/3 to the large party, and, by symmetry, 2/15 to each of the small parties. To see this, we need to see when the marginal contributions of player 1 to any coalition are positive. Recall that there are 6! possible orders of players. Note how, if player 1 arrives first or second in the room in which the coalition is forming, his marginal contribution is zero: the coalition was losing before he arrived and continues to be a losing coalition after his arrival. Similarly, his marginal contribution is also zero if he arrives fifth or sixth to the coalition; indeed, in this case, before he arrives the coalition is already winning, so he adds nothing to it. Thus, only when he arrives third or fourth, which happens a third of the times, does he change the nature of the coalition, from losing to winning. This explains his Shapley value share of 1/3. In this game, the Shapley value payoffs roughly correspond to the proportion of seats that each party has.

Next, consider a third problem in which there are two large parties, while the other four parties are very small. For example, let each of the large parties have 48 seats (say, players 1 and 2), while each of the four small parties has only one seat. Now, the Shapley value payoffs are 0.3 to each of the two large parties, and 0.1 to each of the small ones. To see this, note that the marginal contribution of a small party is only positive when he comes fourth in line, and out of the preceding three parties in the coalition, exactly one of them is a large party, i.e., 72 orders out of the 5! orders in which he is fourth. That is, $(72/5!) \times (1/6) = 1/10$. In this case, the competition between the large parties for the votes of the small parties increases the power of the latter quite significantly, with respect to the proportion of seats that each of them holds.

Finally, consider a fourth problem with two large parties (players 1 and 2) with 46 seats each, one mid-size party (player 3) with 5 seats, and three small parties, each with one seat. First, note that each of the three small parties has become a dummy player: no winning coalition where he belongs becomes losing if he leaves the coalition, and so players 4, 5 and 6 are paid zero by the Shapley value. Now, note that, despite the substantial difference of seats between each large party and the mid-size party, each of them is identical in terms of marginal contributions to a winning coalition. Indeed, for $i = 1, 2, 3$, player i's marginal contribution to a coalition is positive only if he arrives second or third or fourth or fifth (and out of the preceding players in the coalition, exactly one is one of the nondummy players). Note how the Shapley value captures nicely the changes in the allocation of power due to each different political scenario. In this case, the fierce competition between the large parties for the votes of player 3, the swinging party to form a majority, explains the equal share of power among the three.

3.2 The Nucleolus

The nucleolus is another value for TU games. It was introduced in Schmeidler (1969). It is a hard object to define and analyze, but with very nice properties. We shall assume that the game (N, v) is superadditive: for every $S, T \subseteq N, S \cap T = \emptyset$, we have that $v(S \cup T) \geq v(S) + v(T)$.

Consider an efficient allocation x in the game (N, v): $x(N) = v(N)$. For each such x and for each $S \in 2^N \backslash \{N, \emptyset\}$, define the surplus of coalition S at x as: $e_S(x) = x(S) - v(S)$. We shall take this number as an index of the "welfare" of coalition S at x. Define the vector $e(x) \in I\!R^{2^N-2}$ as $e(x) = (e_S(x))_{S \in 2^N \backslash \{N, \emptyset\}}$. This is the vector of all surpluses of the different coalitions at x. Let the vector $e^*(x)$ be a permutation of the entries of $e(x)$ arranged in increasing order.

We shall say that $e(x)$ is leximin superior to $e(y)$ $[e(x) \succ_{lxm} e(y)]$ if $e^*(x)$ is lexicographically superior to $e^*(Y)$, i.e., if there exists $t' + 1 \in \{1, 2, \ldots, 2^{|N|} - 2$ such that $e^*_t(x) = e^*_t(Y)$ for $t = 1, 2, \ldots, t'$ and $e^*_{t'+1}(x) > e^*_{t'+1}(Y)$.

The nucleolus of the game (N, v) is

$$nc(N, v) = \{x \in X(N, v) : \text{there does not exist} y \in X(N, v), \quad e(y) \succ_{lxm} e(x)\}.$$

The nucleolus maximizes the "welfare" of the worst treated coalitions. One can understand it as an application of the Rawlsian social welfare function to a society where each coalition's welfare is evaluated independently.

Example: Let $N = \{1, 2, 3\}$ and consider the following TU game: $v(N) = 42$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, $v(\{i\}) = 0 \forall i \in N$.

Note that the core of this game is empty. Also, the shapley value payoffs are $(9, 14, 19)$.

Let us begin by considering the equal split vector $x = (14, 14, 14)$. Note that

$$e^*(x) = (-12, -2, 8, 14, 14, 14).$$

Here, the worst treated coalition is $\{2, 3\}$. If you were a planner concerned with maximizing the "welfare" of the worst treated coalition, you would like to transfer utility from player

1 to players 2 and 3. For example, consider the vector $y = (4, 24, 14)$, where 10 units have been transferred from player 1 to player 2. Note that

$$e^*(y) = (-12, -2, 4, 8, 14, 24).$$

Thus, we have actually gone in the wrong direction (x is better than y for this planner). However, from x, it seems to make more sense to transfer units from 1 to 3. Consider the vector $z = (4, 14, 24)$, whose associated

$$e^*(z) = (-2, -2, -2, 4, 14, 24).$$

It turns out that $z = \text{nc}(N, v)$.

Next, we shall show a few facts about the nucleolus.

(1) $nc(N, v)$ is individually rational if the game (N, v) is superadditive.

Proof of (1): Suppose not. Let $x \in nc(N, v)$ and suppose there exists $i \in N$ such that $x_i < v(\{i\})$. First, if M is the collection of coalitions S whose surplus is the smallest at x, it must be the case that $i \in S$ for every $S \in M$. To see this claim, suppose not: there exists $S \in M$ and $i \notin S$. Then,

$$x(S \cup \{i\}) - v(S \cup \{i\}) \leq x_i + x(S) - v(S) - v(\{i\}) < x(S) - v(S),$$

a contradiction. Thus, $i \in S$ for every $S \in M$.

Now consider the allocation y: $y_i = x_i + \epsilon$, and for all $j \neq i$, $y_j = x_j - \epsilon/(n-1)$. It should be clear that for every $S \in M$, $e_S(y) > e_S(x)$. Further, choosing $\epsilon > 0$ arbitrarily small, we have that $e_S(y) < e_T(y)$ whenever $S \in M$ and $T \notin M$. Thus, $e(y) \succ_{lxm} e(x)$, which is a contradiction. Q.e.d.

(2) $nc(N, v) \neq \emptyset$ for every game (N, v) where $X_0(N, v) \neq \emptyset$.

Proof of (2): Consider the set $X_0(N, v) = Y_0$. Consider the problem $\max_{x \in Y_0} \min_S x(S) - v(S)$. Note that the function $\min_S x(S) - v(S)$ is continuous in x and that Y_0 is compact. Since the function is continuous and Y_0 is compact, the set of maximizers is nonempty and compact. Denote this by $Y_1 \subseteq Y_0$.

Now write the problem $\max_{x \in Y_1} \min^2_S x(S) - v(S)$, where we denote by $\min^2$ the second worst treated coalitions. By the same arguments, the new set of maximizers Y_2 is nonempty and compact. Since we have a finite number of coalitions, this process can be repeated only a finite number of times. Then, by induction, $nc(N, v) \neq \emptyset$.

(3) $nc(N, v)$ is a singleton.

Proof of (3): Suppose x and $y \in nc(N, v)$, $x \neq y$. Then, $e^*(x) = e^*(y)$. Denote the list of proper coalitions by $S_1, S_2, \ldots, S_m, m = 2^{|N|} - 2$ as arranged in $e^*(x)$. Then, since $x \neq y$, there must exist k to be the first in the order for which $e^*_k(x) \neq e_{S_k}(y)$. Further, it must be the case that $e^*_k(x) < e_{S_k}(y)$. Also, for every $l > k$ in the order, $e^*_l(x) \geq e^*_k(x)$ and $e_{S_l}(y) \geq e^*_k(x)$.

Then, consider the allocation $z = (x + y)/2$. Note that $e^*_h(z) = e^*_h(x)$ for $h < k$, $e^*_k(z) > e^*_k(x)$ and $e^*_l(z) > e^*_k(x)$. Thus, $e(z) \succ_{lxm} e(x)$, which is a contradiction. Q.e.d.

(4) $nc(N, v) \in C(N, v)$ whenever $C(N, v) \neq \emptyset$.

The proof is easy and left as an exercise.

An aplication of the nucleolus to bankruptcy problems: The following example of bankruptcy is taken from the Jewish Talmud. Let E be the estate to be divided, and d_1, d_2 and d_3 the claims of three creditors against the estate E.

(fig. C5) Double entry table. In rows, estates. Over the three columns, the headings $d_1 = 300$, $d_2 = 200$ and $d_3 = 100$. By rows, for $E = 100$, we have 33.3 in each cell; for $E = 200$, we have 75, 75 and 50; in the third row, $E = 300$, 150, 100 and 50.

For centuries, the underlying general principle behind these numbers was unclear. Jewish scholars argued that when $E = 100$, the money was too little to go around; in this case, they argued, every creditor is going to be paid so little that it makes sense to have equal division. If $E = 300$, the estate was exactly half of the sum of the claims. Thus, it makes sense to apply proportional solution and each creditor gets exactly half of her claim. The disturbing fact was that the case $E = 200$ was attributed to an error in transcription.

A separate problem, also found in the Talmud, is described as the contested garment problem. Two men were arguing who owned a garment. One of them said it was his; the other said that half was his. The Talmud gives a clear solution to this problem: the part of the estate conceeded by a claimant is awarded to the other. The rest of the estate, which is contested by both, should be split in half Formally, given a two-agent bankruptcy problem $(E, (d_1, d_2))$, the CG rule is the following:

$$CG(E,(d_1,d_2)) = \left(\max\{0, E-d_2\} + \frac{E - \max\{0, E-d_1\} - \max\{0, E-d_2\}}{2}, \max\{0, E-d_1\} + \frac{E - \max\right.$$

In the above example, this rule assigns the split (0.75,0.25) (which to make matters worse is different from equal or proportional split).

Suppose one is interested in introducing a consistency property in these problems. After all, we understood what the writers of the Talmud wanted to solve two-person problems (the contested garment CG rule), but we have no clue for more than two creditors. The following seems a sensible formulation of consistency:

Let (E, d) be an n-person bankruptcy problem, where $0 \leq E \leq \sum_{i \in N} d_i$ and $d_i \geq 0$. A bankruptcy rule is a function f that assigns to each problem (E, d) a split of the estate $f(E, d)$, i.e., $\sum_{i \in N} f_i(E, d) = E$ and for all $i \in N$, $0 \leq f_i(E, d) \leq d_i$.

A bankruptcy rule f is CG-consistent whenever we have the following: if $f(E, d) = x$ is the n-creditor split, then for every pair i, j, we have that $(x_i, x_j) = CG(x_i + x_j, (d_i, d_j))$.

Define now a coalitional game associated with the bankruptcy problem: $(N, v_{E,d})$, where N is the set of creditors and for every $S \subseteq N$, $v_{E,d}(S) = \max\{0, E - \sum_{i \notin S} d_i\}$.

Theorem (Aumann and Maschler, 1985): There exists a unique rule which is CG-consistent. It is $f(E,d) = nc(N, v_{E,d})$.

Proof: Step 1: We prove that, given a bankruptcy problem (E,d), there exists a unique CG-consistent rule. Suppose not: assume there exist two splits x and y of the estate E that are CG-consistent. That is, for all $i,j \in N$, $(x_i, x_j) = \mathcal{CG}(x_i + x_j, (d_i, d_j))$ and $(y_i, y_j) = \mathcal{CG}(y_i + y_j, (d_i, d_j))$. Note that the CG rule is monotonic, i.e., $\mathcal{CG}(E, (d_i, d_j)) \leq \mathcal{CG}(E', (d_i, d_j))$ whenever $E \leq E'$.

Because $x \neq y$ and $x(N) = y(N) = E$, there exist i and j such that $x_i > y_i$ and $x_j < y_j$. Without loss of generality, suppose $x_i + x_j \geq y_i + y_j$. But then, consistency and monotonicity of the CG rule imply that $(x_i, x_j) \geq (y_i, y_j)$, a contradiction.

Step 2: By notational simplicity, denote the game $(N, v_{E,d})$ by (N, w). Let $x = \mathrm{nc}(N, w)$. By consistency of the nucleolus, we know that for all two-player coalitions $S = \{i, j\}$, $(x_i, x_j) = \mathrm{nc}(S, w_{xS})$. By definition of the nucleolus of this two-player game, we have

$$x_i - w_{xS}(\{i\}) = x_j - w_{xS}(\{j\}).$$

This can be expressed as:

$$x_i = w_{xS}(\{i\}) + \frac{x_i + x_j - w_{xS}(\{i\}) - w_{xS}(\{j\})}{2}.$$

Therefore, we need to check only that

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j\}.$$

To see this, note first that $x \in \mathcal{C}(N, w)$, which implies the core inequalities for all one-person and $(n-1)$-person coalitions, that for all $k \in N$, $0 \leq x_k \leq d_k$.

By the definition of the Davis-maschler reduced game, we have that

$$w_{xS}(\{i\}) = \max_{Q \subseteq N \setminus S} w(\{i\} \cup Q) - x(Q) = w(\{i\} \cup Q^*) - x(Q^*).$$

By the definition of the O'Neill game, we can write that the last expression equals

$$\max\{0, E - d_j - d(N \setminus S \setminus Q^*)\} - x(Q^*) = \max\{-x(Q^*), x_i + x_j - d_j + x(N \setminus S \setminus Q^*) - d(N \setminus S \setminus Q^*)\}.$$

Note that $w_{xS}(\{i\}) \geq 0$ since creditor i always has the option of using $Q^* = \emptyset$. Therefore, the possible values of $w_{xS}(\{i\})$ are:

- When $Q^* = \emptyset$,

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j + x(N \backslash S) - d(N \backslash S) \geq 0.$$

- When $\emptyset \subset Q^* \subset N \backslash S$,

$$w_{xS}(\{i\}) = \max\{-x(Q^*), x_i + x_j - d_j + x(N \backslash S \backslash Q^*) - d(N \backslash S \backslash Q^*)\} \geq 0.$$

- When $Q^* = N \backslash S$,

$$w_{xS}(\{i\}) = \max\{-x(N \backslash S), x_i + x_j - d_j\} \geq 0.$$

Because $0 \leq x \leq d$, it follows that, without loss of generality we can think that either $Q^* = \emptyset$ or $Q^* = N \backslash S$. But then the result follows by considering all possible cases for where the maximum takes place. Q.e.d.

It is also interesting to investigate what kind of non-cooperative bargaining procedures may lead to the nucleolus. Consider the following ones, defined for the class of bankruptcy problems.

Let us start with bilateral bankruptcy problems $(E, (d_i, d_j))$. Let player i make a proposal x, $0 \leq x \leq d$, $x_i + x_j = E$. If player j accepts, the proposal is implemented. If not, a fair coin is tossed. With probability $1/2$, a player will get his best possible outcome and with probability $1/2$ his worst possible outcome. That is, player i will get $\min\{E, d_i\}$ and with probability $1/2$ will get $E - \min\{E, d_j\}$.

Claim: This game's unique subgame perfect equilibrium outcome is the CG rule allocation.

The game $G_1(E, d)$. Let player 1 be one with the highest claim in the multilateral bankruptcy problem (E, d). Player 1 makes a proposal x, $0 \leq x \leq d$, $x(N) = E$. Following the natural protocol, player $i = 2, \ldots, n$ must respond sequentially. If player 2 accepts, he receives x_2 and leaves the game. If he rejects, he receives his share z_2 from the CG rule applied to the problem $(x_1+x_2,(d_1,d_2))$ and leaves the game. Let w^i be player 1's interim share right after he has dealt with player i. Thus, $w^1 = x_1$ and $w^2 = w^1+x_2-\max\{x_2,z_2\}$.

In general, $w^i = w^{i-1}+x_i-\max\{x_i,z_i\}$. If player i accepts, he receives x_i and leaves the game. If he rejects, he receives the share z_i from $\mathcal{CG}(w^{i-1}+x_i,(d_1,d_i))$ and leaves the game. Player 1 ends up with a share w^n and the game ends.

Theorem (Serrano, 1995a): The unique subgame perfect equilibrium outcome of the game $G_1(E,d)$ is $\text{nc}(N, v_{E,d})$.

Note how renegotiation of shares is very important in the noncooperative bargaining game, which must follow the logic of the CG-rule in the reduced bilateral problems.

4 Lecture 4: Advanced Topics

In this fourth lecture we shall spend time on some advanced topics, focusing on recent contributions to two classes of problems, i.e., games with coalitional externalities and games with incomplete information.

4.1 Games with Coalitional Externalities

The notion of the characteristic function assumes that the feasible set of payoffs to a coalition (itw worth in a TU game) is independent of how the complement coalition organizes itself. In some settings, this might be a restrictive assumption (think, for example, of an oligopoly, or a political game played by the political parties in the parliament). This suggest a more general way to represent a cooperative game, which is called the partition function.

We define an n-player TU game in partition function form (Thrall and Lucahs (1963)) as a way to model coalitional externalities:

(N,v) is the game, where

$N = \{1,\dots,n\}$ is a finite set of players,

$v(S, \Pi)$ is the worth of coalition S when the coalition structure is $\Pi = \{S, T, \ldots, R\}$, a partition of N.

Assume: $v(\{\emptyset\}, \Pi) = 0$ for every partition Π.

Let us illustrate this definition with the following example:

Example: A free rider problem.

$N = \{1, 2, 3\}$,

$v(N, \{N, \{\emptyset\}\}) = 24$,

$v(\{1, 2\}, \{\{1, 2\}, \{3\}, \{\emptyset\}\}) = 12$,

$v(\{1, 3\}, \{\{1, 3\}, \{2\}, \{\emptyset\}\}) = 13$,

$v(\{2, 3\}, \{\{2, 3\}, \{1\}, \{\emptyset\}\}) = 14$,

$v(\{i\}, \{\{i\}, \{j, k\}, \{\emptyset\}\}) = 9$ for all i, j, k,

$v(\{i\}, \{\{1\}, \{2\}, \{3\}, \{\emptyset\}\}) = 0$ for all i.

Marginal contributions and externalities: Suppose player i leaves coalition S to join T. One can then define

Total effect:

$$v(S, \{S, T, \Pi_{-S,-T}\}) - v(S_{-i}, \{S_{-i}, T_{+i}, \Pi_{-S,-T}\}),$$

which can be decomposed into two:

Marginal contribution: $v(S, \{S, T, \Pi_{-S,-T}\}) -$ $v(S_{-i}, \{S_{-i}, \{i\}, T, \Pi_{-S,-T}\})$

and Externality: $v(S_{-i}, \{S_{-i}, \{i\}, T, \Pi_{-S,-T}\}) -$ $v(S_{-i}, \{S_{-i}, T_{+i}, \Pi_{-S,-T}\})$.

Consider the following axiom:

Weak marginality: the payoff to player $i \in N$ in the solution must depend on the entire vector of total effects (including both marginal contributions and externalities)

$$(v(S, \{S, T, \Pi_{-S,-T}\}) - v(S_{-i}, \{S_{-i}, T_{+i}, \Pi_{-S,-T}\}))_{S,T,\Pi}.$$

Then, the following result is found, perhaps surprisingly, after what we knew from characteristic functions:

A large class of **linear** and even **non-linear** solutions satisfy efficiency, anonymity and weak marginality.

A related axiom one could consider is this:

Monotonicity: Consider two games (N, v) and (N, w). If $\delta_i(v) \geq \delta_i(w)$ (total effects), the solution must pay i in v at least as much as in w.

And the third axiom we shall consider is this:

(Intrinsic) Marginality: the payoff to player $i \in N$ in the solution must depend only on the intrinsic marginal contributions, not on the externalities:

$$(v(S, \Pi) - v(S_{-i}, \{S_{-i}, \{i\}, \Pi_{-S}\}))_{S,\Pi}.$$

Then, one can prove the following result:

Theorem (de Clippel and Serrano, 2008a): There exists a unique solution σ^* satisfying efficiency, anonymity and marginality:

$$\sigma_i^*(v) = \mathrm{Sh}_i(N, \hat{v}),$$

where

$$\hat{v}(S) := v(S, \{S, \{j\}_{j \notin S}\}).$$

Call this the "externality-free" value.

For an illustration of the result, consider the following example:

Example: An asymmetric free rider.

$v(N,\{N,\{\emptyset\}\})=24,$

$v(\{1,2\},\{\{1,2\},\{3\},\{\emptyset\}\})=12,$

$v(\{1,3\},\{\{1,3\},\{2\},\{\emptyset\}\})=13,$

$v(\{2,3\},\{\{2,3\},\{1\},\{\emptyset\}\})=14,$

$v(\{1\},\{\{1\},\{2,3\},\{\emptyset\}\})=9,$

$v(\{i\},\{\{i\},\{j,k\},\{\emptyset\}\})=0$ for $i=2,3,$

$v(\{i\},\{\{1\},\{2\},\{3\},\{\emptyset\}\})=0$ for all i.

Externality-free value:

$$\sigma^*(v)=(7.5,8,8.5).$$

Bounds to payoffs given by monotonicity:

$$\sigma_1(v)\in[7.5,10.5],$$

$$\sigma_2(v)\in[6.5,8],$$

$$\sigma_3(v)\in[7,8.5].$$

Thus, with respect to the externality-free value, a Pigouvian-like planner concerned with these axioms (efficiency, anonymity, and monotonicity) would think of transfering surplus from players 2 and 3 to player 1, who is the only one that can benefit from an externality, when players 2 and 3 cooperate. In general, these bounds provide the maximum tax/subsidy that one can allow in order to still respect the axioms.

In a different contribution, Macho-Staedler, Perez-Castrillo and Wettstein (2007) define the average approach to partition functions, by taking suitable averages of the different underlying characteristic functions, and explore the corresponding extension of the Shapley value.

Finally, taking a strategic approach closer to the Nash program, Maskin (2003) investigates a solution to the corresponding coalition formation problem in these contexts with externalities. His results are complemented by those in de Clippel and Serrano (2008b).

See also Macho-Stadler, Perez-Castrillo and Wettstein (2006) for a different extensive form that models coalition formation.

4.2 Games with Incomplete Information

The seminal contribution in the literature of cooperative games with asymmetric information is Wilson (1978), who makes two proposals on how to extend the core to these settings. The analysis is performed at the interim stage, i.e., that in which agents have private information (each of them knows his or her type, but not necessarily the types of the other agents). Wilson's proposals are based on exogenous processes of communication within coalitions, and he concentrates on the two polar opposite cases. The first one, in which no communication takes place, leads to the coarse core: coalitions must block when it is common knowledge among all players in the coalition that there exists an improvement in terms of interim expected utility for all the types. This yields the largest possible core: allocations excluded from the coarse core are safely excluded, as even if agents could not communicate, they should figure out that they could block. The other extreme allows for all possible forms of communication, and it leads to the fine core. While Wilson proves nonemptiness of the coarse core in convex exchange economies, the fine core is often empty. When one adds incentive compatibility constraints, even the coarse core may be empty, as shown in Vohra (1999) and Forges, Mertens and Vohra (2002).

As it turns out, the core convergence theorem may also fail for the coarse core, as shown in Serrano, Vohra and Volij (2001). We introduce necessary notation to work in economies with asymmetric information.

An exchange economy E:

N: finite set of consumers.

Ω: finite set of states of the world.

$\mathbb{R}^l_+$: consumption set for each consumer in each state.

$x_i : \Omega \mapsto \mathbb{R}^l_+$: consumption plan.

X_i: set of consumption plans for consumer i.

$e_i \in X_i$: endowment of agent i.

$u_i : \mathbb{R}^l_+ \times \Omega \mapsto \mathbb{R}$: i's Bernoulli utility function.

P_i: i's private information, partition of Ω.

$P_i(\omega)$: element of P_i which contains ω.

μ_i: i's prior probability measure on Ω.

Assumption: for each $A \in P_i$, $\mu_i(A) > 0$.

Consumer i's conditional expected utility at $A \in P_i$:

$$U_i(x_i \mid A) = \sum_{\omega \in A} \mu_i(\omega \mid A) u_i(x_i(\omega), \omega).$$

Feasible Allocation for coalition $S \subseteq N$: $(x_i)_{i \in S} \in \prod_i X_i$ such that

$$\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ for all } \omega \in \Omega.$$

A_S: set of feasible allocations for S.

The *meet* of the partitions $(P_i)_{i \in S}$ is the finest partition of Ω that is coarser than each P_i, $i \in S$, and it is denoted by $P_S = \wedge_{i \in S} P_i$. An event E is said to be *common knowledge among the members of* S *at* ω if $P_S(\omega) \subseteq E$.

Coarse objection to $x \in A_N$: $\exists y \in A_S$ and an event $E \subseteq P_S$ such that

$$U_i(y_i \mid A) > U_i(x_i \mid A) \quad \forall i \in S, \forall A \in P_i$$

such that $A \subseteq E$.

Coarse core: set of $x \in A_N$ with no coarse objection.

Price equilibrium:

Let $p = (p(\omega))_{\omega \in \Omega}$ denote a vector of state-contingent market prices where $p(\omega) \in \mathbb{R}^l$ for $\omega \in \Omega$. Let $\triangle$ denote the unit simplex in $\mathbb{R}^{l \times |\Omega|}$. For $A \in P_i$, let

$$\begin{aligned} \gamma_i(p \mid A) = \quad & \{x_i(A) \in X_i(A) \mid \\ & \textstyle\sum_{\omega \in A} p(\omega) \cdot x_i(\omega) \leq \sum_{\omega \in A} p(\omega) \cdot e_i(\omega)\}. \end{aligned}$$

A *constrained market equilibrium* is defined as $(x, p) \in A_N \times \triangle$ such that for every $i \in N$ and $A \in P_i$,

$$x_i(A) \in \arg \max_{\gamma_i(p|A)} U_i(. \mid A).$$

Constrained market equilibria satisfy several properties that are analogous to those of Walrasian equilibria.

(i) An equilibrium allocation belongs to the coarse core.

(ii) A replication of an equilibrium allocation is an equilibrium allocation of the corresponding replicated economy.

(iii) The converse of (ii) also holds with strictly concave Bernoulli utility functions.

Property P. *Suppose (x, p) is a price equilibrium and there exists $\omega \in \Omega$ and a consumer i such that $\{\omega\} \in P_i$. Then*

$$x_i(\omega) \in \arg \max_{\gamma_i(p|\{\omega\})} u_i(., \omega).$$

(That is, a fully informed consumer maximizes ex post utility over ex post budget constraint).

Example 1.

- $N = \{1, 2\}$, $\Omega = \{s, t\}$
- $P_1 = (\{s, t\})$ and $P_2 = (\{s\}, \{t\})$

- $u_i((a,b),\omega) = (ab)^{1/4}$ for $i = 1,2$ and for $\omega = s,t$
- $e_1(s) = e_1(t) = (0,24)$ and $e_2(s) = e_2(t) = (24,0)$
- $\mu_i(s) = \mu_i(t) = \frac{1}{2}$, for $i = 1,2$.

In Example 1, if a price equilibrium concept satisfies Property P, unique equilibrium: $(\bar{x}, \bar{p})$, where

$$\bar{x}_1(s) = \bar{x}_2(s) = \bar{x}_1(t) = \bar{x}_2(t) = (12,12),$$

$$\bar{p}(s) = \bar{p}(t) = (1/4, 1/4).$$

Furthermore, for any integer m, $(\bar{x}^m, \bar{p})$ is the unique price equilibrium in E^m.

Also, $\bar{x}^m$ belongs to the coarse core.

However, in spite of the above extensions of standard results, we have the following:

Proposition (Serrano, Vohra and Volij, 2001): Let E be the economy defined in Example 1. There exists an allocation x in E whose replica x^m is in the coarse core of E^m for all m, such that x cannot be supported as an equilibrium satisfying property P.

Proof. Consider the following allocation x:

$$x_1(s) = (9,9), \qquad x_1(t) = (16,16)$$

$$x_2(s) = (15,15), \qquad x_2(t) = (8,8).$$

The authors show the robustness of the result to a number of core concepts. However, in more recent work, Kamishiro and Serrano (2011) establish some positive core convergence results for core concepts and quasilinear economies, in which the information transmission within coalitions is endogenous (through the Bayesian equilibria of some communication mechanism, as modelled in Serrano and Vohra (2007)). See also de Clippel (2007) for a different core convergence result when information revelation takes place through a competitive bidding process with outside market makers. Two recent surveys of results in this area are Forges, Minelli and Vohra (2002) and Forges and Serrano (2013).

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