

Lloyd Shapley's Matching and Game Theory*

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Abstract

This is a survey of Lloyd Shapley's contributions to matching theory and game theory in general, starting with the work that inspired the Swedish Academy to award Shapley and Roth the 2012 Nobel Memorial Prize in Economic Sciences.

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“Lloyd Shapley, whom I consider to be the greatest game theorist of all time”

(Robert J. Aumann, 2005 Nobel Lecture).

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1 Introduction

This is a survey of the numerous fundamental contributions that Lloyd S. Shapley has made to game theory, with a special emphasis on his path-breaking results in matching theory. The latter were the basis of the 2012 Nobel Memorial Prize in Economic Sciences, awarded to Lloyd Shapley and Alvin Roth. Roth's contributions are also fundamental, both in terms of the refinement of the theoretical models and predictions and in their implementation in practice, which has led to the important new area of market design. I will leave the detailed description of Roth's work to others, and concentrate here on Shapley's rich legacy.

Shapley's trajectory in game theory is good proof of the use of different approaches. His contributions include the concept of stable matchings, a value for coalitional games, the perfect "folk" theorem of repeated games, the model of stochastic games, or potential games, to name just a few. When you ask Lloyd Shapley, he defines himself as a mathematician. His approach to problems is abstract, not distracted by the details of each specific example.

In analyzing a game, one can begin by writing down its extensive form or game tree and understand the role that the information held by players at each point in time, the timing of moves, and the credibility of threats play in the solution. All these considerations feature prominently in the perfect "folk" theorem for repeated games (Aumann and Shapley (1976), Rubinstein (1979)).

One can take a step back from the extensive form, and specify the strategies available to each player in the strategic or normal form of the game. One poses the solution to the game then as a theory of behavior: say, player 1 is

supposed to choose her first strategy and player 2 his third strategy, perhaps because after analyzing the strategic interaction between both, each of them figures out that, given what the other player is likely to do, it is in her/his own interest to choose the strategy specified, or perhaps a player chooses a strategy as a safety device to protect her payoff no matter what the other player could do. Already early on, Shapley (1953b) works in the normal form and analyzes strategically the model of a stochastic game.

One can take yet another step back from details, ignore the strategies that players can potentially choose, and concentrate on the payoffs or sets of payoffs that are available to each player or set of players, as the characteristic function form does. Or one can just specify the sets of agents and the preferences of each agent over the set of relevant outcomes, as done in a matching problem. Given those primitives (e.g., the characteristic function form of a game or the preferences in a matching problem), one can require properties of desirable solutions, such as efficiency or stability and see what those requirements imply. The nonemptiness of stable allocations proved by Gale and Shapley (1962) for marriage problems, or the value of characteristic function games with transferable utility (TU games) shown in Shapley (1953a) are two very different, but salient examples of the axiomatic methodology. Shapley was one of its first adopters in game theory. The method first defines a domain of relevant problems, then poses a list of desirable properties, and finally tries to identify whether such properties can lead to a solution. Affirmative answers to this quest yield positive results, while answers in the negative lead to impossibilities.

As stands out in the preceding paragraphs, Shapley's approach to game

theory is integrated, never discarding a tool that can prove useful to shed light on the problem at hand. Different methods (i.e., axiomatic and strategic) and different models (e.g., games in extensive form and matching problems) are all in his tool box. In this regard, while it is true that the axiomatic method has been applied in game theory mostly in problems involving pairs or coalitions of players, there is in principle no reason to exclude its use from the analysis of strategic or extensive forms, as in axiomatic and epistemic approaches to equilibrium in games. Therefore, identifying axiomatic game theory with the cooperative part of the discipline (i.e., the one where groups of players are the actors) is a misunderstanding. By the same token, it is a misunderstanding as well to identify strategic game theory with the noncooperative counterpart (i.e., the one in which individual players are the protagonists), as shown for example by the very active area of research known as coalition formation.

As confirmed by the path of the discipline in recent years, it is my expectation that, following Shapley's steps, among other giants, the borderlines between the noncooperative and cooperative parts of game theory become completely fluid, as in the end they together can only enrich our conclusions about a problem by looking at it from different angles. Nash (1953) pioneered this view, which led to the birth of the Nash program (see Serrano (2008) for a survey). Indeed, it is remarkable that Shapley (1953a) also devotes its last section to a bargaining understanding of his value.

This survey is organized as follows. Section 2 describes Shapley's contributions to matching theory. Section 3 goes over his work in general games. For ease of reading, each section is divided in subsections. Section 4 concludes.

2 Shapley's Matching Theory

Matching theory has become a leading area in economic theory in order to deal with the difficulties presented by indivisibilities in the allocation of resources. Calculus has to be dropped as a tool, and instead one should turn to linear programming and combinatorics. To a large extent, the area follows Shapley's path-breaking contributions.

2.1 Matching: Marriage and Roommate Problems

This area was initiated in a paper written by David Gale and Lloyd Shapley fifty years ago. The most basic problem proposed in Gale and Shapley (1962) is the marriage problem. Consider a set of n men and a set of n women (the fact that we assume the same number of men and women is not essential for the results). Each man has a complete and transitive preference ordering over the set of women, and each woman has a complete and transitive preference ordering over the set of men. Assume there are no indifferences.

We say that a matching of men and women is stable if there do not exist a man and a woman who prefer each other to their current partners in the matching. If such a pair exists, we say that the matching is unstable. One nontrivial question is whether, regardless of the preference orderings, one can always find a stable matching. The answer, in the affirmative, is provided by Gale and Shapley (1962):

Theorem 1: *There always exists a stable matching in the marriage problem.*

The proof of this theorem is constructive, based on the now rightly famous deferred-acceptance algorithm, which is explained as follows. Let one of the sides (e.g., the women) make the proposals. Let each woman make a matching proposal to her most preferred man. Each man who receives more than one proposal rejects all proposals but the one that is best for him among the set of proposals received. Importantly, each man does not accept yet the proposal that he holds on to, but waits until the end of the algorithm (deferred-acceptance). In the second stage, the rejected women make a proposal to their second-ranked men. Again, those men who now have more than two proposals reject all but the one they prefer the most among the set of received proposals, and so on. The algorithm ends when no man rejects any proposal, at which point all proposals are accepted and a matching results. This happens after a finite number of steps, since no woman makes a proposal twice to the same man.

It is easy to see that the resulting matching is stable. Consider man m_i and woman w_j , who are not matched in the matching in question. Note that for m_i and w_j not to have been matched, either (1) woman w_j never proposed to man m_i indicating that she actually prefers the partner that she ended up with better than m_i , or (2) woman w_j did propose to man m_i but he rejected her, holding on to a better prospect. Therefore, in either case, such a pair cannot improve upon the matching. The matching is stable.

Incidentally, the nonemptiness of the set of stable matchings is not to be taken for granted, and it depends on the problem. Gale and Shapley (1962) already noted the difficulties that one encounters in the related roommates problem. Suppose a set of n agents, where n is even, is trying to solve the

problem of finding a roommate, and consider the following preferences (P_i means “preferred by i ”):

$$a_2 P_1 a_3 P_1 a_4;$$

$$a_3 P_2 a_1 P_2 a_4;$$

$$a_1 P_3 a_2 P_3 a_4;$$

and any preference for agent a_4 . The reader can easily check that there are no stable matchings, given the above preference cycle of the first three agents.

Going back to the marriage problem, it is also important to note that the two deferred-acceptance algorithms (the women-proposing and the men-proposing) yield the women’s optimal stable matching and the men’s optimal stable matching, respectively.

We say that a stable matching is optimal for one side if every agent on that side is matched with her(his) top-ranked partner within all stable matchings.

From the definition, it should be clear that, if the optimal stable matching exists for one side, it is unique. To see this, recall our assumption that all preferences are strict. The following result can be shown by induction:

Theorem 2: *In the marriage problem, the deferred-acceptance algorithm where one side proposes yields the optimal stable matching for that side.*

Gale and Shapley (1962) also consider the more general college admissions problem, in which each college can admit up to a quota of students. Even though much work has taken place in recent years, Roth and Sotomayor (1990) remains still today a classic reference that systematizes the different problems in the area.

2.2 Assignment: Housing Problems

There are other problems in which one of the sides “does not have preferences.” These are problems in which certain indivisible objects must be assigned to a group of individuals. Examples are houses among a group of consumers, offices among a collection of professors in an economics department, or kidneys among a set of patients.

The seminal paper on assignment markets is due to Lloyd Shapley and Herbert Scarf (Shapley and Scarf (1974)). In it, a housing market is modelled as a collection of objects—the houses—, each of them owned by an agent. Each agent has preferences over the houses and wants only one house.

Shapley and Scarf (1974) propose an algorithm, known as top-trading cycles, that always gives an assignment of houses to agents that can be supported by competitive prices in equilibrium (the algorithm was suggested to the authors by David Gale). From the strong version of the first welfare theorem, it follows that the resulting assignment is in the core, which is the generalization to this setting of stable allocations.

We say that an assignment of houses to agents is in the core if there does not exist a coalition S of agents that can redistribute the houses they own in such a way that all of them strictly prefer the house from the reallocation over the house they are getting in the assignment.

Shapley and Scarf’s (1974) main result follows:

Theorem 3: *The core of any housing problem is nonempty.*

We focus on the proof based on the top-trading cycles algorithm, which

works as follows. Each agent points to the house that he or she prefers the most. Each cycle of agents so formed is asked to exchange their houses within the cycle, as indicated by their wishes, and removed. All other agents remain still with their owned houses. In the second stage, the remaining agents point to their top house among this remaining set, new trading cycles are formed, and so on. The process ends after a finite number of steps with an assignment of houses to agents. Choosing any sequence of positive prices that is decreasing over time yields competitive prices that support the assignment as a competitive equilibrium allocation.

The algorithm gives a unique prediction if preferences are strict (if no individual is indifferent between any two houses), as shown in Roth and Postlewaite (1977). The core, however, may contain more than one assignment, even in this case. With indifferences, the top-trading cycles algorithm retains its core stability properties, but uniqueness is lost (there are multiple competitive equilibrium allocations).

Shapley and Scarf (1974) provide a separate proof that the core of the housing market is nonempty by showing that the associated nontransferable utility (NTU) cooperative game is balanced. They then appeal to a theorem of Scarf (1967). The condition of balancedness will be discussed in the next section.

2.3 The Assignment Game: Trading Indivisible Goods for Money

Although prices have been mentioned in the previous subsection, their presence is only virtual: they simply act as bookkeeping devices in the computa-

tion of a competitive equilibrium, but in truth, all trade takes place without the use of money. In contrast, following Shapley and Shubik (1971), we consider in this subsection a model of exchange of indivisible goods for money. There are two sides to the market, buyers and sellers. Each seller wants to buy one unit of an indivisible good, and each buyer wants to buy at most one unit. The units might differ in costs of production and each buyer may have different valuations for each of the units. Utility functions are quasilinear in money: the utility function of each buyer is her valuation for the good minus the price she pays, while the seller's is the price he charges minus production cost.

The effective coalitions in this game are two-player coalitions consisting of one buyer and one seller. Indeed, the key ones are the ones that generate the highest surplus when trading. Generically, this will lead to a unique assignment of goods to buyers (if there are ties, one has to be a bit more careful in such an assignment, but it will not affect the results). It is easy also to calculate the highest overall surplus, which is the worth of the coalition of all traders (the worth of other large coalitions is irrelevant, as all gains from trade are realized by the buyer-seller pairs). Shapley and Shubik (1971) show the following result:

Theorem4: *The core of the assignment game with money is nonempty. Moreover, it coincides with the set of competitive equilibrium allocations of the economy.*

The authors prove this result by constructing the dual of the problem

and appealing to linear programming techniques. It is also interesting that, as in the case of the Gale-Shapley marriage problem, the core has a similar structure. Namely, there is an extreme point in the core in which all sellers are best treated and all buyers are worst treated (this corresponds to the prices of all goods being as high as possible). Similarly, the other extreme point of the core is the best for all buyers and the worst for all sellers.

3 Shapley's General Game Theory

In this section, we shall go over the many contributions that Shapley has made to cooperative games, their connections with the theory of general equilibrium, and noncooperative games.

3.1 Cooperative Games: The Shapley Value

The following is the definition of a game (N, v) in characteristic function form with transferable utility. Let N be a finite set of players. For each nonempty subset S of N , let $v(S)$ be the worth of S . The function v is called the characteristic function, and S is called a coalition. The worth of S is interpreted as the total surplus that S will create should its members decide to cooperate. Can we think of a way to split the worth of the Grand Coalition $v(N)$ among the players in N ?

To answer this question, Shapley (1953a) proposes the following four axioms. First, efficiency (i.e., the sum of individual payoffs must be exactly $v(N)$). Second, anonymity (i.e., the names of the players should not matter; in particular, if two players are perfect substitutes in the characteristic

function, their payoff must be the same). Third, additivity (since utility is transferable, the payoff to each player from a characteristic function that results from the sum of two characteristic functions should be the sum of payoffs received in each of the two characteristic functions). Fourth, dummy player (i.e., if a player contributes nothing to each coalition, his payoff must be zero).

How many solutions or values (i.e., functions that assign to each (N, v) a single payoff vector) are there that satisfy the four stated axioms? Shapley (1953a) shows that there is only one, which we call today the Shapley value:

Theorem 5: *There is a unique solution or value that satisfies efficiency, anonymity, additivity, and dummy player. It is given by the following formula. For each player i :*

$$\text{Sh}_i(N, v) = \sum_{S \subseteq N} \frac{(|S| - 1)! (|N| - |S|)!}{|N|!} [v(S) - v(S \setminus \{i\})].$$

The expression $[v(S) - v(S \setminus \{i\})]$ is called the marginal contribution of player i to coalition S . Thus, the Shapley value pays to each player the average of all her marginal contributions. The weight that each marginal contribution receives in the average corresponds to all possible orderings of players being equally likely.

The Shapley value is, along with the core, the central solution in cooperative game theory (see Serrano (2009) for a survey). Shapley's remarkable result is a tribute to the marginalist principle, of old tradition in economic theory: each agent should be paid according to his or her marginal productiv-

ity. Yet, with the possible exception of the dummy player axiom, no explicit mention of marginalism is made in the Shapley axioms, which makes the result extremely surprising. For a point of comparison, the reader is invited to consult Young (1985), who, using marginalism as an axiom, but dropping additivity, also obtains the Shapley value as the unique implication of his axioms; see Macho-Stadler, Pérez-Castrillo, and Wettstein (2007), de Clippel and Serrano (2008), and the references therein for related recent discussions that apply to games with coalitional externalities (those in which the worth of a coalition may depend on the actions of the complement coalition). Winter (2002) and Monderer and Samet (2002) survey the concept of the Shapley value and some of its variants, while Mertens (2002b) illustrates its economic applications.

One of the important early applications of the Shapley value is found in Shapley and Shubik (1954). This paper considers a committee and tries to measure the power of each member of the committee. The authors construct a characteristic function of a simple game (Shapley (1962)), in which the worth of a coalition is 1 if it is a winning coalition, and 0 otherwise. The Shapley value of this game awards to each player the fraction of orders in which he or she is pivotal, i.e., the probability of being pivotal when all orders are equally likely. The value so calculated leads to what today is called the Shapley-Shubik power index, and the authors illustrate its use in different applications, including the power of permanent versus nonpermanent members of the United Nations Security Council. Within this fascinating area of quantitative political theory, the use of the Shapley-Shubik power index has become quite prevalent. One of its main alternative indices is the

Banzhaf power index, which counts the number of times a player turns a losing coalition into a winning one. The Banzhaf index lacked an axiomatic analysis supporting it, and Dubey and Shapley (1979) provided it, enhancing the comparison between the two power indices.

The assumption of uniform orders in the Shapley-Shubik power index was relaxed in Owen (1971), Shapley (1977), and Owen and Shapley (1989), with the consideration of ideologies, leading to the Shapley-Owen asymmetric power index. Another assumption implicit in the Shapley-Shubik power index is that voters are independent and cannot influence each other. Building on the concept of authority organizations in Shapley (1994), Hu and Shapley (2003a, b) relax the assumption and formulate an equilibrium authority distribution from a power-in/power-out mechanism, which turns out to be quite close to the invariant distribution of a Markov chain.

If one drops the assumption of transferable utility, one is led to considering the more general class of games in characteristic function with nontransferable utility (NTU games). In this case, the numeric worth of coalition S is replaced by a feasible set of payoffs, a subset of $\mathbb{R}^{|S|}$, that coalition S can achieve should its members cooperate. In a deep contribution related to Harsanyi (1963), Shapley (1969) proposes a way to locally linearize the boundary of the feasible set of each coalition, hence creating an associated TU-like game. The utility transfers are not necessarily one-to-one across different players: one needs to allow λ -transfers, where λ is the normal to the hyperplane tangent to the boundary of the feasible set for the Grand Coalition, capturing the marginal tradeoffs of utility at that point. To this game, the Shapley value construction is applied, leading to what is called the

Shapley λ -transfer value. This is one of the possible generalizations of the Shapley value to NTU games, though not the only one (see McLean (2002) for a survey).

Shapley (1969) also speaks to utility theory, because of its way to handle the issue of interpersonal utility comparisons. In another recent contribution to utility theory, Baucells and Shapley (2008) formalize and establish a claim first found in Shapley and Shubik (1982, p. 66). The problem is how to aggregate individual utility functions into a social utility function. Using the extended Pareto principle (if two disjoint coalitions prefer one alternative over another, so does their union), the authors show that even if the aggregate preferences are incomplete for coalitions (assuming that individual preferences are complete, though), one can construct a utilitarian social utility function. Baucells and Shapley (2008, p. 330) acknowledge that this is a first step in Shapley's long-term desire to axiomatize the assumption of transferable utility.

3.2 Cooperative Games: Other Solution Concepts

Shapley's contributions to the theory of the core, the other leading solution in cooperative games, are of central importance. Section 2 already dealt with some of the ideas, when focusing on matching problems, but this subsection describes his achievements in developing the core of general games and economies.

Although the idea of coalitional recontracting and its consequent final settlements can be found in Edgeworth (1881), Shapley and Shubik (1966, footnote 2) attribute the concept of the core in its current game-theoretic

formulation to Gillies and Shapley. Indeed, this was part of Gillies' doctoral dissertation and was also the subject of an informal conference held at Princeton, both dated in 1953 (see also Gillies (1959), to whom Aumann (1961) attributes the idea of the core, though restricted to TU games).

The first fundamental contribution of Shapley's to the general theory of the core concerns its nonemptiness in the class of TU games. In two independently written papers, Bondareva (1963) and Shapley (1967) prove that the core of a TU game is nonempty if and only if the game is balanced. To define balancedness, one needs to define a set of balancing weights. Let w_S denote the weight given to coalition S . The collection of weights $(w_S)_{S \subseteq N}$ is said to be balanced if for every player i , $\sum_{S, i \in S} w_S = 1$. That is, if one thinks that each player has one unit of time available, he or she plans to allocate it among the different coalitions S of which he or she is a member according to the fractions w_S .

The TU game (N, v) is said to be balanced whenever, for every collection of balancing weights $(w_S)_{S \subseteq N}$, one has that $\sum_S w_S v(S) \leq v(N)$.

The Bondareva-Shapley theorem can be then stated as follows:

Theorem 6: *The core of the TU game (N, v) is nonempty if and only if the game (N, v) is balanced.*

Balancedness means that the sum of worths of subcoalitions, when weighted by the time-sharing arrangement specified in a set of balancing weights, does not exceed the worth of the Grand Coalition N . The concept can be extended, using the same logic, to NTU games, and in fact, Scarf (1967) shows

that balancedness is a sufficient condition to guarantee the nonemptiness of the core of NTU games, although it is no longer necessary (see, however, Predtetchinski and Herings (2004)). Shapley (1973) provides an alternative proof of Scarf’s theorem, building connections with the mathematical theory of fixed points, in particular with Brouwer’s fixed point theorem, and Shapley and Vohra (1991) provide an elementary proof of Scarf’s theorem based on Kakutani’s fixed point theorem. The reader is referred to Kannai (1992) for a survey of the use of balancedness in the theory of the core.

An important class of balanced TU games is that of convex games. Convex games are those in which there are “increasing returns” to cooperation. Specifically, the marginal contribution of any player i to any coalition S is no greater than his or her marginal contribution to T whenever S is a subset of T . It is important to point out that, in general, the Shapley value payoff profile need not be in the core when this is nonempty, exhibiting a discrepancy between the goals stated in the Shapley axioms and coalitional stability. However, Shapley (1971) shows the following remarkable result:

Theorem 7: *The core of a convex TU game is exactly the convex hull of the $n!$ vectors of marginal contributions. Therefore, the Shapley value payoff profile in convex TU games is in the core.*

Indeed, for convex games, the Shapley value occupies a central position in the core, being its center of gravity, as the weights given to the $n!$ vectors of marginal contributions are all equal (i.e., $1/n!$).

In a beautiful series of papers with Martin Shubik, Shapley studies the

cores and approximate cores of different classes of games and economies. In Shapley and Shubik (1966), it is shown that in nonconvex quasilinear exchange economies, approximate cores become nonempty if one admits a sufficiently large number of agents. Shapley and Shubik (1969a) considers economies with externalities, and shows that the core is nonempty if all externalities are positive, while it might be empty in the presence of negative externalities. Shapley and Shubik (1969b) studies the class of market games, i.e., exchange economies in which agents have continuous, concave, and quasilinear utility functions, and shows that each market game is equivalent to a totally balanced TU game. (A TU game is said to be totally balanced when the game and all its subgames are balanced; a subgame is a projection of the characteristic function to a subset of players and all its subsets.) Therefore, the core of market games is always nonempty, and the paper provides useful transformations to go from markets to games and vice versa.

In two papers with Michael Maschler and Bezalel Peleg, Shapley considers other important solution concepts, including the bargaining set, the kernel and the nucleolus; Maschler (1992) is a useful survey. The bargaining set checks for the credibility of coalitional objections in the process of recontracting: only those coalitional objections to an allocation that do not have a counterobjection are listened to, hence leading to a larger set than the core. Unlike the core, it can be shown that the bargaining set is always nonempty. In the class of convex games (defined above), Maschler, Peleg, and Shapley (1972) shows that the bargaining set actually coincides with the core, rendering all objections used in core recontracting credible.

The kernel is another set-valued solution concept, in which each pair of

players i and j finds itself in “equilibrium”, in the sense that *ceteris paribus*, the payoff increase that each of them (say, i) could get by threatening the other (say, j) with leaving the Grand Coalition and going along with i ’s most preferred coalition that excludes j is the same. One can make the kernel independent of the interpersonal utility comparisons implied in this statement by viewing it as a set of points in which all players are in a bilateral bargaining equilibrium (see Serrano (1997)). In convex games, Maschler, Peleg, and Shapley (1972) shows that the kernel reduces to a single point, which is the nucleolus. The nucleolus is a value different from the Shapley value, and it is defined as the payoff in which the welfare of the worst treated coalition is maximized, in a somewhat Rawlsian tradition in which each coalition enters as an entity of social welfare. In particular, the nucleolus is always in the core when this is nonempty: unlike the Shapley value, it is a single-valued solution that never conflicts with coalitional stability.

We state together the two main results of Maschler, Peleg, and Shapley (1972):

Theorem 8: *(i) The bargaining set of a convex TU game coincides exactly with its core. (ii) The kernel of a convex TU game coincides exactly with its nucleolus.*

Finally, Maschler, Peleg, and Shapley (1979) provide a number of elegant geometric results for all these solution concepts, enhancing in particular their bargaining content.

3.3 Large Games and Perfectly Competitive Economies

An important area in economic theory has dealt with the connections between the predictions that game theory makes when there are many players and those given by the theory of general economic equilibrium. This is the so-called equivalence principle (see, e.g., Aumann (1987, Section 1960-1970)), which, roughly speaking, states that under certain conditions the predictions of game theory in large enough games should approximate or coincide with the set of competitive allocations. This principle is methodologically reassuring, as one would expect the science of strategic interactions –game theory– to converge to perfect competition, if individual agents become negligible. Naturally, things are not so simple. Today we know that, although there are many illustrations of the principle, there are also robust exceptions, or instances of slow convergence (e.g., Shapley (1975)). On the whole, the results in this area have contributed to a better understanding of perfect competition from very different angles.

In the context of economies with an atomless continuum of traders, The collaboration between Robert Aumann and Lloyd Shapley led to another classic. Aumann and Shapley (1974) shows the equivalence between the Shapley value payoffs and the Walrasian equilibrium allocations.

Theorem 9: *In quasilinear economies with an atomless continuum of agents, the competitive allocations yield a payoff distribution equal to the Shapley value payoff.*

This and other results were partially advanced in Shapley and Shubik

(1969c), which provides examples of approximation between perfect competition, core, and value allocations for large enough games. Moreover, convergence is established for a class of market games. The reader is referred to the surveys by Hart (2002) for values of competitive economies, and Neyman (2002) for values of games with infinitely-many players.

Shapley and Shubik also derived results in the equivalence principle using concepts in noncooperative game theory. An early result is an oligopolistic model whose Nash equilibria converge to perfect competition (Shapley and Shubik (1967)).

3.4 Noncooperative Games

One of the most important classes of noncooperative games studied is that of repeated games. A repeated game is the repetition of a given game G , which is called the stage game. The theory of repeated games offers a wide array of predictions, often leading to cooperative behavior, supported in equilibrium by strategies that specify the punishments to those who deviate from the specified behavior and the rewards to those that administer the proper punishment to the deviators. The result known as the “folk” theorem for repeated games states that there is a large multiplicity of equilibrium outcomes associated with the repeated game if the future matters sufficiently. Every feasible payoff (i.e., in the convex hull of payoffs of G) that is individually rational can arise as the average payoff of a Nash equilibrium in the infinite repetition of G if discounting is small enough.

One issue is the credibility of the punishments. That is, following a deviation from a Nash equilibrium strategy profile, is it in the interest of the

punishers to enforce the prespecified penal code? One way to address this question is by changing the solution concept, from Nash equilibrium to subgame perfect equilibrium, and Aumann and Shapley (1976) and Rubinstein (1979) give independently two different versions of an affirmative answer to this question. This is known today as the perfect “folk” theorem (undiscounted version):

Theorem 10: *Every feasible and individually rational payoff in G can be supported by a subgame perfect equilibrium of the undiscounted infinitely repeated game (using the limit of the means or the overtaking criterion for the payoffs in the infinitely repeated game).*

The version of this theorem with discounting is due to Fudenberg and Maskin (1986). For a survey of the rich theory of repeated games, the reader is referred to Mailath and Samuelson (2006), a recent comprehensive monograph.

A stochastic game is a more general model than that of a repeated game. Suppose there are one or more players and multiple states or positions. A state corresponds to a potentially different action set and payoff function for each player. The game is played over a finite or infinite number of stages. When the game is at a given state, players choose actions simultaneously and receive a stage payoff that depends on the action profile chosen and on the state. Both the action profile chosen and the state determine a probability distribution detailing how likely it is to move to each different state. The payoff in the stochastic game is the average of the payoffs received in each

stage. Note how the model reduces to a Markov decision problem if there is only one player, and to a repeated game if there is only one state. Shapley (1953b) introduces this model and analyzes stationary strategies in the case of two-player zero-sum games. Stationary strategies are history-independent, but they allow action choices that do depend on the state. Shapley (1953b) characterizes the minimax value of the stochastic game and shows its existence supported by stationary strategies. Stochastic games are a challenging class of games, but impressive progress has taken place in its analysis (see Mertens (2002a) and Vieille (2002) for surveys).

The problem of how to learn to play a Nash equilibrium has received much attention in recent years (see Fudenberg and Levine (1998) and Young (1998)). Historically, one of the adjustment processes studied in the early years of game theory was that of fictitious play, proposed by Brown (1951). Fictitious play asks each player to start playing the game choosing his or her first action arbitrarily. In subsequent stages, each player is asked to best-respond to the mixed strategy that corresponds to the relative frequency of actions chosen by the opponent. The first results were encouraging: Robinson (1951) showed its convergence to equilibrium in two-person zero-sum games, and Miyasawa (1961) in the class of non-zero-sum games where both players have only two actions. However, Shapley (1964) provides a striking example of a two-player non-zero-game with three actions in which fictitious play does not converge its dynamics lead to expanding orbits farther and farther away from the unique Nash equilibrium. Monderer and Shapley (1996a) presents the class of potential games, those in which the payoff function of each player is strategically equivalent to a single function, which is called

the potential of the game. Potential games always have a pure-strategy equilibrium. In Monderer and Shapley (1996b), it is shown that fictitious play always converges to the Nash equilibrium set. Potential games are also relevant in a wide class of evolutionary processes (see Weibull (1995) and Vega-Redondo (1996)).

Finally, in a model that complements their market games, but using the noncooperative approach, Shapley and Shubik (1977) begins the analysis of what today we call strategic market games. The motivation is to propose a model of trade in which prices are endogenously determined by the actions of traders (who, therefore, are not price-takers). The authors propose a model in which one commodity is treated as “money” or “cash” that will buy the other goods. The setup is an exchange economy, in which each trader is endowed with a bundle of commodities, including an amount of money. The traders submit all the amounts they have of the nonmoney commodities to a number of trading posts, but they retain a claim to their endowed amounts when writing down their budget sets. Each trader submits bids of the money commodity for each of the other commodities, and the price of each nonmonetary commodity is determined as a function of the submitted bids. Trade takes place as a function of the submitted bids and the computed prices. It is important to point out that, unlike the Walrasian model of exchange, the outcome is defined in and out of equilibrium, and that agents can affect the prices by changing their bids. The game is solved by resorting to Nash equilibrium. Strategic market games have been analyzed to provide noncooperative foundations of perfect competition and also to suggest a way to justify the use of money in economic exchange. Dubey (1994) and Giraud

(2003) provide surveys of the field.

4 Concluding Remarks

The previous sections should have served the reader to gain a wide perspective of the many important contributions of Lloyd Shapley to matching and game theory. I cannot think of a better way to close the survey than using the two concluding remarks from Gale and Shapley (1962), Shapley's own favorite among all his papers, with the perspective of the fifty years elapsed since its publication.

1. "The reader who has followed us this far has doubtless noticed a certain trend in our discussion. In making the special assumptions needed in order to analyze our problem mathematically, we necessarily moved further away from the original college admission question, and eventually in discussing the marriage problem, we abandoned reality altogether and entered the world of mathematical make-believe. The practical-minded reader may rightfully ask whether any contribution has been made toward an actual solution of the original problem. Even a rough answer to this question would require going into matters which are nonmathematical, and such discussion would be out of place in a journal of mathematics. It is our opinion, however, that some of the ideas introduced here might usefully be applied to certain phases of the admissions problem."

Indeed, the whole enterprise of market design is an amazing confirmation of this opinion, and the Swedish Academy has confirmed it with the 2012 Nobel Prize award to "the theory of stable allocations and the practice of

market design.” As for their second remark, which follows, it is a beautiful description of the appropriate use Of mathematics in game theory and economics, a great way to characterize Shapley’s elegant work and mathematical mind:

2. “Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which nonmathematicians believe mathematics to be concerned with. Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with “a head for figures”, or that they “know a lot of formulas”. At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter. What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematical, and the reason that your friends and ours cannot understand mathematics is not because they have no head for figures, but because they are unable to achieve the degree of concentration required to follow a mod-

erately involved sequence of inferences. This observation will hardly be news to those engaged in the teaching of mathematics, but it may not be so readily accepted by people outside of the profession. For them the foregoing may serve as a useful illustration.”

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