Four Lectures on the Nucleolus and the Kernel
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Roberto Serrano

Department of Economics, Brown university

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Preliminaries

We begin by defining a non-transferable (NTU) game in coalitional form (N, V), where $N = \{1, 2, \ldots, n\}$ is a finite set of players, and for every $S \subseteq N, S \neq \emptyset$, $V(S) \subseteq I\!R^{|S|}, V(S) \neq \emptyset$. In the sequel and abusing notation, we shall identify the cardinality of a set with its lower case representation. Thus, $|S| = s$.

Two subclasses of NTU games are:

1. pure bargaining problems: The pair (U, d) is a pure bargaining problem, where $U = (V(N)) \subseteq I\!R^n$ is the feasible utility set and $d \in I\!R^n$ is the disagreement or threat point. Therefore, for all $S \neq N$, $V(S) \subseteq \Pi_{i \in S} V(\{i\})$. That is, intermediate size coalitions are powerless.

2. Transferable utility (TU) games: For every $S \subseteq N$, there exists a real number $v(S)$ such that
$$V(S) : \{x \in I\!R^s : \sum_{i \in S} x_i \leq v(S)\}.$$
That is, there exists a numeraire good that allows transfers of utility from player to player at a one-to-one rate. For this subclass of problems, we shall use the following notational conventions.

    1. Denote by 2^N the set of all subsets of N and let $v(S) : 2^N \mapsto I\!R$. Then, we speak of the pair (N, v) as the TU game.

    2. Given a vector $x \in I\!R^n$, we denote by $x(S) = \sum_{i \in S} x_i$ and by x^S the projection of x to the subspace corresponding to the players in S.

    3. We shall denote by $X(N, v)$ the Pareto frontier of the game (N, v), i.e.,
    $$X(N, v) = \{x \in I\!R^n : x(N) = v(N)\}.$$

    4. We shall denote by $X_0(N, v)$ the individually rational and efficient payoff set in the game (N, v), i.e.,
    $$X_0(N, v) = \{x \in X(N, v) : x_i \geq v(\{i\}) \quad \forall i \in N\}.$$

One last preliminary observation. What we will call nucleolus and kernel here is what the literature has called prenucleolus and prekernel. These are the true solution concepts, which do not impose individual rationality (as opposed to nucleolus and kernel properly speaking). In many economic applications, the distinction is irrelevant, because the prenucleolus and prekernel will turn out to be individually rational.

1 The Nucleolus: Definition and Properties

The nucleolus is a solution concept for the class of TU games (some generalizations to the class of NTU games exist, but they are somewhat problematic). It is a somewhat mysterious solution concept. The nucleolus is a hard object to define and analyze, but with very nice properties.

Consider $x \in X(N, v)$. For each such x and for each $S \in 2^N \backslash \{N, \emptyset\}$, define the excess of coalition S at x as: $e_S(x) = x(S) - v(S)$. We shall take this number as an index of the "welfare" of coalition S at x. Define the vector $e(x) \in I\!R^{2^N-2}$ as $e(x) = (e_S(x))_{S \in 2^N \backslash \{N, \emptyset\}}$. This is the vector of all excesses of the different coalitions at x. Let the vector $e^*(x)$ be a permutation of the entries of $e(x)$ arranged in increasing order.

We shall say that $e(x)$ is leximin superior to $e(y)$ $[e(x) \succ_{lxm} e(y)]$ if $e^*(x)$ is lexicographically superior to $e^*(Y)$, i.e., if there exists $t'+1 \in \{1, 2, \ldots, 2^n - 2$ such that $e_t^*(x) = e_t^*(y)$ for $t = 1, 2, \ldots, t'$ and $e^*_{t'+1}(x) > e^*_{t'+1}(y)$.

Definition (Schmeidler, 1969): The (pre)nucleolus of the game (N, v) is

$$\text{nc}(N, v) = \{x \in X(N, v) : \text{there does not exist} y \in X(N, v), \quad e(y) \succ_{lxm} e(x)\}.$$

The nucleolus maximizes recursively the "welfare" of the worst treated coalitions. One can understand it as an application of the Rawlsian social welfare function to a society where each coalition's welfare is evaluated independently.

Example 1: Let $N = \{1, 2, 3\}$ and consider the following TU game: $v(N) = 42$, $v(\{1, 2\}) = 20$, $v(\{1, 3\}) = 30$, $v(\{2, 3\}) = 40$, $v(\{i\}) = 0 \forall i \in N$. Let us begin by considering the equal split vector $x = (14, 14, 14)$. Note that

$$e^*(x) = (-12, -2, 8, 14, 14, 14).$$

Here, the worst treated coalition is $\{2, 3\}$. If you were a planner concerned with maximizing the "welfare" of the worst treated coalition, you would like to transfer utility from player 1 to players 2 and 3. For example, consider the vector $y = (4, 24, 14)$, where 10 units have been transferred from player 1 to player 2. Note that

$$e^*(y) = (-12, -2, 4, 8, 14, 24).$$

Thus, we have actually gone in the wrong direction (x is better than y for this planner). However, from x, it seems to make more sense to transfer units from 1 to 3. Consider the vector $z = (4, 14, 24)$, whose associated

$$e^*(z) = (-2, -2, -2, 4, 14, 24).$$

It turns out that $z = \text{nc}(N, v)$.

Next we show some properties of the nucleolus.

(1) Individual rationality: $\text{nc}(N, v) \in X_0(N, v)$ if the game (N, v) is superadditive: for every $S, T \subseteq N, S \cap T = \emptyset$, we have that $v(S \cup T) \geq v(S) + v(T)$.

Proof of (1): Suppose not. Let $x \in \text{nc}(N, v)$ and suppose there exists $j \in N$ such that $x_j < v(\{j\})$ [call i to the player for whom the individual excess is smallest: $x_i - v(\{i\}) = \min_{j \in N} x_j - v(\{j\})$].

First, if M_1 is the collection of coalitions S whose excess is the smallest at x, it must be the case that $i \in S$ for every $S \in M_1$. To see this, suppose not: there exists $S \in M_1$ and $i \notin S$. Then,

$$x(S \cup \{i\}) - v(S \cup \{i\}) \leq x_i + x(S) - v(S) - v(\{i\}) < x(S) - v(S),$$

a contradiction. Thus, $i \in S$ for every $S \in M_1$.

Now consider the allocation y: $y_i = x_i + \epsilon$, and for all $j \neq i$, $y_j = x_j - \epsilon/(n-1)$. It should be clear that for every $S \in M_1$, $e_S(y) > e_S(x)$. Further, choosing $\epsilon > 0$ arbitrarily small, we have that $e_S(y) < e_T(y)$ whenever $S \in M_1$ and $T \notin M_1$. Thus, $e(y) \succ_{lxm} e(x)$, which is a contradiction. Q.e.d.

(2) Non-emptiness: $\text{nc}(N, v) \neq \emptyset$.

Proof of (2): (We present a proof assuming that (N, v) is superadditive. If not, one must show first that the nucleolus lives in a compact set.) Consider the set $X_0(N, v) = Y_0$. Consider the problem $\max_{x \in Y_0} \min_S x(S) - v(S)$. Note that the function $\min_S x(S) - v(S)$ is continuous in x and that Y_0 is compact. Since the function is continuous and Y_0 is compact, the set of maximizers is non-empty and compact. Denote this by $Y_1 \subseteq Y_0$.

Now write the problem $\max_{x \in Y_1} \min^2_S x(S) - v(S)$, where we denote by $\min^2$ the second worst treated coalitions. By the same arguments, the new set of maximizers Y_2 is non-empty and compact. Since we have a finite number of coalitions, this process can be repeated only a finite number of times. Then, by induction, $\text{nc}(N, v) \neq \emptyset$. Q.e.d.

(3) Uniqueness: $\text{nc}(N, v)$ is a singleton.

Proof of (3): Suppose x and $y \in \text{nc}(N, v)$, $x \neq y$. Then, $e^*(x) = e^*(y)$. Denote the list of proper coalitions by $S_1, S_2, \ldots, S_m, m = 2^n - 2$ as arranged in $e^*(x)$. Then, since $x \neq y$, there must exist k to be the first in the order for which $e^*_k(x) \neq e_{S_k}(y)$. Further, it must be the case that $e^*_k(x) < e_{S_k}(y)$. Also, for every $l > k$ in the order, $e^*_l(x) \geq e^*_k(x)$ and $e_{S_l}(y) \geq e^*_k(x)$.

Then, consider the allocation $z = (x + y)/2$. Note that $e^*_h(z) = e^*_h(x)$ for $h < k$, $e^*_k(z) > e^*_k(x)$ and $e^*_l(z) > e^*_k(x)$. Thus, $e(z) \succ_{lxm} e(x)$, which is a contradiction. Q.e.d.

(4) Core selection: $\text{nc}(N, v) \in \mathcal{C}(N, v)$ whenever $\mathcal{C}(N, v) \neq \emptyset$.

The proof is easy and left as an exercise.

(5) Consistency: Let $x = \text{nc}(N, v)$ and define the Davis-Maschler reduced game (S, v_{xS}) for coalition S as follows. Let $v_{xS}(S) = v(N) - x(N \backslash S)$; and for every $T \subset S, T \neq \emptyset$, $v_{xS}(T) = \max_{Q \subseteq N \backslash S} v(T \cup Q) - x(Q)$. Then, $x^S = \text{nc}(S, v_{xS})$ for every $S \subseteq N$. This is a property of internal consistency of a solution. Remarkably, this exact property is shared by other solution concepts (such as the core, the bargaining set and, as we shall see, the kernel).

(6) Covariance: Let (N, v) be a game, $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}^n$. Construct the game (N, w), where for every $S \subseteq N$, $w(S) = \alpha v(S) + \beta(S)$. Then, $\text{nc}(N, w) = \alpha \text{nc}(N, v) + \beta$. That is, it is invariant to positive affine transformations that preserve the TU property of the game.

(7) Anonymity: Let $\pi : N \mapsto N$ be a bijection. Then, $\text{nc}(\pi(N, v)) = \pi(\text{nc}(N, v))$. That is, players' names do not matter.

Two interesting characterizations of the nucleolus have been found. One is based on balanced collections of coalitions and the other on well-known axioms.

A collection of coalitions $M = \{S_1, \ldots, S_k\}$ is balanced if one can find coefficients $\lambda_1, \ldots, \lambda_k$ associated with each coalition, $0 \leq \lambda_{k'} \leq 1$ for $1 \leq k' \leq k$ such that for all $i \in N$, $\sum_{i \in S_{k'} \in M} \lambda_{k'} = 1$.

Fix a payoff vector $x \in X(N, v)$. Let $M_1(x)$ be the collection of coalitions with the smallest excess at x; let $M_2(x)$ be the collection of coalitions with the second smallest excess at x, and so on.

Theorem (Kohlberg, 1971): Fix an arbitrary TU game (N, v). The payoff vector x is $\text{nc}(N, v)$ if and only if for all $k \geq 1$, $\cup_{k'=1}^{k} M_{k'}(x)$ are balanced collections.

In particular, it follows from this theorem that, for every player, $M_1(\text{nc}(N, v))$ must contain at least one coalition that includes this player and at least one coalition that excludes him.

Theorem (Sobolev, 1975): Consider the class of all TU games. There exists a unique non-empty single-valued solution satisfying covariance, anonymity and consistency: it is $\text{nc}(N, v)$.

it is interesting to compare these axioms to those behind other solutions. For example, the Shapley value satisfies all except consistency (although if one replaces the Davis/Maschler reduced game with the one introduced by hart/Mas-Colell, the Shapley value is characterized with the other four axioms). On the other hand, the core satisfies all but non-emptiness and single-valuedness. In the class where it is non-empty (balanced games), the only difference between the core and the nucleolus is whether one wants to insist on single-valuedness.

2 The Nucleolus: Applications and Bargaining

There are by now a few applications of the nucleolus to different problems. In most cases, its use has been motivated as an alternative to the Shapley value, the other single-valued solution to TU Games. These applications include cost allocation problems (as in the Birmingham airport runways, or in different network problems) or surplus sharing problems, one of whose particular cases is bankruptcy. We shall concentrate on the aplication of the nucleolus to bankruptcy problems.

The following example of bankruptcy is taken from the Talmud. Let E be the estate to be divided, and d_1, d_2 and d_3 the claims of three creditors against the estate E.

(fig. 1) Double entry table. In rows, estates. Over the three columns, the headings $d_1 = 300$, $d_2 = 200$ and $d_3 = 100$. By rows, for $E = 100$, we have 33.3 in each cell; for $E = 200$, we have 75, 75 and 50; in the third row, $E = 300$, 150, 100 and 50.

For centuries, the underlying general principle behind these numbers was unclear. Jewish scholars argued that when $E = 100$, the money was too little to go around; in this case, they argued, every creditor is going to be paid so little that it makes sense to have equal division. If $E = 300$, the estate was exactly half of the sum of the claims. Thus, it makes sense to apply the proportional solution and each creditor gets exactly half of her claim. The disturbing fact was that the case $E = 200$ was attributed to an error in transcription.

A separate problem, also found in the Talmud, is described as the contested garment problem.. Two men were arguing who owned a garment. One of them said it was his; the other said that half was his. The Talmud gives a clear solution to this problem: the part of the estate conceeded by a claimant is awarded to the other. The rest of the estate, which is contested by both, should be split in half Formally, given a two-agent bankruptcy problem $(E, (d_1, d_2))$, the CG rule is the following:

$$\mathcal{CG}(E, (d_1, d_2)) = \begin{pmatrix} \max\{0, E - d_2\} + \frac{E - \max\{0, E-d_1\} - \max\{0, E-d_2\}}{2} \\ \max\{0, E - d_1\} + \frac{E - \max\{0, E-d_1\} - \max\{0, E-d_2\}}{2} \end{pmatrix}$$

In the above example, this rule assigns the split $(0.75, 0.25)$, which to make matters worse is different from equal or proportional split.

Suppose one is interested in introducing a consistency property in these problems. After all, we understood how the writers of the Talmud wanted to solve two-person problems (the contested garment CG rule), but we have no clue for more than two creditors. The following seems a sensible formulation of consistency:

Let (E, d) be an n-person bankruptcy problem, where $0 \leq E \leq d(N)$ and $d_i \geq 0$. A bankruptcy rule is a function f that assigns to each problem (E, d) a split of the estate $f(E, d)$, i.e., $\sum_{i \in N} f_i(E, d) = E$ and for all $i \in N$, $0 \leq f_i(E, d) \leq d_i$.

A bankruptcy rule f is CG-consistent whenever we have the following: if $f(E, d) = x$ is the n-creditor split, then for every pair i, j, we have that $(x_i, x_j) = \mathcal{CG}(x_i + x_j, (d_i, d_j))$.

Define now a coalitional game associated with the bankruptcy problem: $(N, v_{E,d})$, where N is the set of creditors and for every $S \subseteq N$, $v_{E,d}(S) = \max\{0, E - d(N \backslash S)\}$.

Theorem (Aumann and Maschler, 1985): There exists a unique rule which is CG-consistent. It is $f(E, d) = \text{nc}(N, v_{E,d})$.

Proof: Step 1: We prove that, given a bankruptcy problem (E, d), there exists a unique CG-consistent rule. Suppose not: assume there exist two splits x and y of the estate

E that are CG-consistent. That is, for all $i, j \in N$, $(x_i, x_j) = \mathcal{CG}(x_i + x_j, (d_i, d_j))$ and $(y_i, y_j) = \mathcal{CG}(y_i + y_j, (d_i, d_j))$. Note that the CG rule is monotonic, i.e., $\mathcal{CG}(E, (d_i, d_j)) \leq \mathcal{CG}(E', (d_i, d_j))$ whenever $E \leq E'$.

Because $x \neq y$ and $x(N) = y(N) = E$, there exist i and j such that $x_i > y_i$ and $x_j < y_j$. Without loss of generality, suppose $x_i + x_j \geq y_i + y_j$. But then, consistency and monotonicity of the CG rule imply that $(x_i, x_j) \geq (y_i, y_j)$, a contradiction.

Step 2: By notational simplicity, denote the game $(N, v_{E,d})$ by (N, w). Let $x = \text{nc}(N, w)$. By consistency of the nucleolus, we know that for all two-player coalitions $S = \{i, j\}$, $(x_i, x_j) = \text{nc}(S, w_{xS})$. By definition of the nucleolus of this two-player game, we have

$$x_i - w_{xS}(\{i\}) = x_j - w_{xS}(\{j\}).$$

This can be expressed as:

$$x_i = w_{xS}(\{i\}) + \frac{x_i + x_j - w_{xS}(\{i\}) - w_{xS}(\{j\})}{2}.$$

Therefore, we need to check only that

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j\}.$$

To see this, note first that $x \in \mathcal{C}(N, w)$, which implies the core inequalities for all one-person and $(n-1)$-person coalitions, that for all $k \in N$, $0 \leq x_k \leq d_k$.

By the definition of the Davis-maschler reduced game, we have that

$$w_{xS}(\{i\}) = \max_{Q \subseteq N \setminus S} w(\{i\} \cup Q) - x(Q) = w(\{i\} \cup Q^*) - x(Q^*).$$

By the definition of the O'Neill game, we can write that the last expression equals

$$\max\{0, E - d_j - d(N \setminus S \setminus Q^*)\} - x(Q^*) = \max\{-x(Q^*), x_i + x_j - d_j + x(N \setminus S \setminus Q^*) - d(N \setminus S \setminus Q^*)\}.$$

Note that $w_{xS}(\{i\}) \geq 0$ since creditor i always has the option of using $Q^* = \emptyset$. Therefore, the possible values of $w_{xS}(\{i\})$ are:

- When $Q^* = \emptyset$,

$$w_{xS}(\{i\}) = \max\{0, x_i + x_j - d_j + x(N\backslash S) - d(N\backslash S) \geq 0.$$

- When $\emptyset \subset Q^* \subset N\backslash S$,

$$w_{xS}(\{i\}) = \max\{-x(Q^*), x_i + x_j - d_j + x(N\backslash S\backslash Q^*) - d(N\backslash S\backslash Q^*)\} \geq 0.$$

- When $Q^* = N\backslash S$,

$$w_{xS}(\{i\}) = \max\{-x(N\backslash S), x_i + x_j - d_j\} \geq 0.$$

Because $0 \leq x \leq d$, it follows that, without loss of generality we can think that either $Q^* = \emptyset$ or $Q^* = N\backslash S$. But then the result follows by considering all possible cases for where the maximum takes place. Q.e.d.

It is also interesting to investigate what kind of non-cooperative bargaining procedures may lead to the nucleolus. Consider the following ones, defined for the class of bankruptcy problems.

Let us start with bilateral bankruptcy problems $(E, (d_i, d_j))$. Let player i make a proposal x, $0 \leq x \leq d$, $x_i + x_j = E$. If player j accepts, the proposal is implemented. If not, a fair coin is tossed. With probability 1/2, a player will get his best possible outcome and with probability 1/2 his worst possible outcome. That is, player i will get $\min\{E, d_i\}$ and with probability 1/2 will get $E - \min\{E, d_j\}$.

Claim: This game's unique subgame perfect equilibrium outcome is the CG rule allocation.

The game $G_1(E, d)$. Let player 1 be one with the highest claim in the multilateral bankruptcy problem (E, d). Player 1 makes a proposal x, $0 \leq x \leq d$, $x(N) = E$. Following the natural protocol, player $i = 2, \ldots, n$ must respond sequentially. If player 2 accepts, he receives x_2 and leaves the game. If he rejects, he receives his share z_2 from the CG rule applied to the problem $(x_1 + x_2, (d_1, d_2))$ and leaves the game. Let w^i be player 1's interim share right after he has dealt with player i. Thus, $w^1 = x_1$ and $w^2 = w^1 + x_2 - \max\{x_2, z_2\}$.

In general, $w^i = w^{i-1} + x_i - \max\{x_i, z_i\}$. If player i accepts, he receives x_i and leaves the game. If he rejects, he receives the share z_i from $\mathcal{CG}(w^{i-1} + x_i, (d_1, d_i))$ and leaves the game. Player 1 ends up with a share w^n and the game ends.

Theorem (Serrano, 1995): The unique subgame perfect equilibrium outcome of the game $G_1(E, d)$ is $\text{nc}(N, v_{E,d})$.

3 The Kernel: Definition and properties

The kernel is a set-valued solution concept that has been criticized for being based on interpersonal comparisons of utility. As we shall see, this criticism is not entirely valid. Although its mathematical properties are well-known for the class of TU Games, its generalizations to the class of NTU games exist. We will present one of them, which we will call the nash set. In this lecture, we begin by presenting the kernel of TU Games.

Let (N, v) be a TU game. For each pair of players $k, l \in N$, define the surplus of player k against l at the payoff vector x as follows:

$$s_{kl}(x) = \max_{k \in S, l \notin S} \{v(S) - x(S)\}.$$

This is the maximum utility that player k can expect to get by departing from x without the consent of player l.

Definition (Davis and Maschler, 1965): The (pre)kernel of the game (N, v) is the set

$$\mathcal{K}(N, v) = \{x \in X(N, v) : s_{kl}(x) = s_{lk}(x) \quad \forall k, l \in N\}.$$

Thus, at a kernel payoff, all players are in a sort of "bilateral equilibrium", in the sense that the threats to each other are equalized. The definition seems to involve interpersonal utility comparisons. We will revisit this issue, though. Next we present some properties of the kernel.

(1) individual rationality: If the game (N, v) is superadditive, $\mathcal{K}(N, v) \in X_0(N, v)$.

Proof of (1): Let $x \in \mathcal{K}(N, v)$ and suppose that $x_i < v(\{i\})$. Recall that the set $M_1(x)$ is the set of coalitions that, at x, receive the smallest excess.

We begin by showing that $i \in S$ if $S \in M_1(x)$. Consider first any coalition $T \subset N \backslash \{i\}$. Then, $T \notin M_1(x)$ because:

$$e_T(x) = x(T) - v(T) > x(T \cup \{i\}) - v(T) - v(\{i\}) \geq x(T \cup \{i\}) - v(T \cup \{i\}) = e_{T \cup \{i\}}(x).$$

(A weak form of superadditivity –0 monotonicity– was used in the last inequality). Moreover, $T = N \backslash \{i\} \notin M_1(x)$ either, because:

$$e_{N \backslash \{i\}}(x) = x(N \backslash \{i\}) - v(N \backslash \{i\}) > x(N) - v(N \backslash \{i\}) - v(\{i\}) \geq x(N) - v(N) = 0,$$

whereas for example $e_{\{i\}}(x) < 0$.

Therefore, if $S \in M_1(x)$, $i \in S$. Since $N \backslash \{i\} \notin M_1(x)$, there exists $j \in N$ such that there exists a coalition $S \in M_1(x)$ such that $j \notin S$. But then, $s_{ij}(x) > s_{ji}(x)$, which is a contradiction. Q.e.d.

(2) Non-emptiness: For all TU games (N, v), $\mathcal{K}(N, v) \neq \emptyset$.

Proof of (2): Several proofs are available. One uses that the relation $s_{kl}(x) > s_{lk}(x)$ is transitive and appeals to the KKM lemma. A second proof shows that $\text{nc}(N, v) \in \mathcal{K}(N, v)$. (Recall the observation we made after Kohlberg's theorem).

(3) Core bisection: If $x \in \mathcal{K}(N, v) \cap \mathcal{C}(N, v)$, forany $i, j \in N$, $i \neq j$, and fixing $x^{N \backslash \{i,j\}}$, the point (x_i, x_j) bisects the intersection of the core and the transfer line between players i and j (the "bargaining range" between players i and j). Thus, it would appear that the kernel does not depend on interpersonal utility comparisons, at least those kernel payoffs that are also in the core.

(4) Consistency. To see this, note that the maximization involved in the definition of the surplus can be separated into two stages, one using players out of the reduced game and a second one, using players in the reduced game.

(5) Converse consistency. Let $x \in X(N, v)$. If for every two-player reduced game (S, v_{xS}) we have that $x^S \in \mathcal{K}(S, v_{xS})$, then $x \in \mathcal{K}(N, v)$.

(6) Covariance.

(7) Equal treatment: If $x \in \mathcal{K}(N, v)$, $x_i = x_j$ whenever players i and j are substitutes. This means that for every $S \subseteq N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Theorem (Peleg, 1986): Over the class of all TU games, there exists a unique solution satisfying non-emptiness, Pareto efficiency, covariance, equal treatment, consistency and converse consistency. It is $\mathcal{K}(N, v)$.

4 The Kernel and the Nash Set

In this lecture, developing further the comments we made after the core bisection property, I would like to reinterpret the kernel in a way that makes it independent of interpersonal utility comparisons. Recall the basic equations of the kernel:

$$s_{kl}(x) = s_{lk}(x),$$

which can be rewritten as:

$$v_{x\{kl\}}(\{k\}) - x_k = v_{x\{kl\}}(\{l\}) - x_l,$$

or:

$$x_k = \frac{1}{2} v_{x\{kl\}}(\{k\}) + \frac{1}{2}[x_k + x_l - v_{x\{kl\}}(\{l\})].$$

That is, first reinterpret the two-player reduced game as a bargaining problem between players k and l, where the "pie" to be divided is $x_k + x_l$ and where the threat point is $(v_{x\{kl\}}(\{k\}), v_{x\{kl\}}(\{l\}))$. The kernel is then the set of payoffs where every pair of players splits in half this pie (when modified by the outside options embodied in the threat point). This is a generalization of the insight of Maschler, Peleg and Shapley (1979), in seeing the kernel as payoffs where certain bilateral ranges are split in half. The advantage is that this is a fact inherently linked to the definition of the kernel, and quite independent of the core.

Let us see a couple of examples:

Example 1: Consider the TU game (N,v), where $N=\{1,2,3\}$, and $v(\{i\})=0 \forall i \in N$, $v(\{1,2\})=4$, $v(\{1,3\})=3$, $v(\{2,3\})=2$, $v(N)=6$. Then, $\mathcal{K}(N,v)=(3,2,1)$. Figure 2 represents the reduced problems faced by the pairs of players $(1,2)$, $(1,3)$ and $(2,3)$. The payoff in the kernel splits in half the "available surplus" to each pair determined by the threat point, which is a function of the payoff awarded to the third player.

Example 2: Consider again the example we saw on the first lecture: a 3-player 0-normalized game where $v(\{1,2\})=20$, $v(\{1,3\})=30$, $v(\{2,3\})=40$, $v(N)=42$. Again, the kernel is a singleton: $\mathcal{K}(N,v)=(4,14,24)$. Figure 3 represents the three two-player reduced problems, where the threat points now lie outside of the feasible set.

In fact, this reinterpretation of the kernel lends itself easily to a generalization to NTU games.

Consider the class of smooth NTU games, where the boundary of $V(N)$ admits a differentiable representation $g(x)=0$. Denote by $g_i(x)$ the partial derivative of g with respect to x_i at the payoff vector x.

Let us introduce two more properties:

(8) Scale invariance: Consider a NTU game (N,V), let $\alpha \in I\!\!R^n_{++}$ and $\beta \in I\!\!R^n$. Transform player i's payoff function from x_i into $\alpha_i x_i + \beta_i$ and call the resulting NTU game (N,W). A solution σ satisfies scale invariance if $\sigma(N,W)=\alpha\sigma(N,V)+\beta$.

(9) Local independence: Let (N,V) and (N,V') be two games with all other things equal except $V(N) \neq V'(N)$. Let x be efficient in both games, i.e., $g(x)=0$ and $g'(x)=0$. Furthermore, let the gradient of $g(x)$ be parallel to the gradient of $g'(x)$. A solution σ satisfies local independence if, whenever $x \in \sigma(N,v)$, $x \in \sigma(N,V')$.

Theorem (Serrano and Shimomura, 1998): over the class of smooth NTU games, there exists a unique solution satisfying non-emptiness for two-player games, Pareto efficiency, scale invariance, equal treatment in TU Games, local independence, consistency and converse consistency. It is the Nash set:

$$\mathcal{N}(N,v) = \{x \in V(N) : g(x) = 0 \quad \text{and} \quad \forall k \neq l, g_k(x)[V_{x\{kl\}}(\{k\}) - x_k] = g_l(x)[V_{x\{kl\}}(\{l\}) - x_l]\}.$$

That is, at a payoff of the Nash set, the elasticity of the payoff difference relative to the threat point for each pair of players is 1. Defining $dx_l/dx_k = g_k(\cdot)/g_l(\cdot)$, we have:

$$\frac{dx_l}{dx_k}(x)\frac{x_k - V_{x\{kl\}}}{x_l - V_{x\{kl\}}} = 1.$$

A graphic way to represent this is that the relevant bargaining range is split in half. By the relevant range, we mean the segment of the tangent plane truncated at the coordinates of the threat point. Figure 4 represents this when the threat point is in the feasible set, and figure 5 does when it is outside of it.

Example 3: Consider the following two person non-convex pure bargaining problem. Suppose two bargainers are negotiating over how to split two dollars and the consent of both is needed to split any pie. Suppose player i's utility function for $i = 1, 2$ is the following: $u(x_i) = x_i$ if i's share $x_i \leq 1$, while $u(x_i) = 4x_i - 3$ otherwise. Then, $\mathcal{N}$ consists of three points for this case:

$$\mathcal{N}(N,V) = \{(1,1), (5/2, 5/8), (5/8, 5/2)\}.$$

(See Figure 6). That is, three possible splits of the pie are prescribed: equal division (the problem is symmetric) and two others where the risk loving agent receives 11/8, while the risk neutral one gets 5/8.

The Nash set coincides with the kernel for TU games, and with the Nash solution when applied to convex pure bargaining problems. Clearly, much more research is needed on this solution concept. Although existence is a problem in the general class of NTU games, it will be interesting to uncover restricted classes of games where the Nash set is non-empty. Also, it should be tested in applications as a natural generalization of the Nash solution to contexts where convexity is not assumed and where coalitions play a role. Finally, the equivalence question is not trivial (it is known that the Nash set is not contained in the bargaining set).

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