# Lesson 13
# Duopoly



Roberto Serrano and Allan M. Feldman



Version C

## *1. Introduction*

In this lesson, we study market structures that lie between perfect competition and monopoly. As before we assume, at least in most of this lesson, that there is one homogeneous good which is the same no matter who makes it. We assume everyone has perfect information about the good and its price. In our discussion of monopoly, we assumed there were barriers to entry which preserved the monopolist's position. In this lesson. we also assume there are barriers to entry which prevent other firms from entering the market. However, we now assume there are already two (or more) firms in the market.

An *oligopoly* is a market with just a few firms. For instance, the market for cell phone service in our part of the U.S. is currently dominated by Verizon Wireless, ATT, and Sprint, a total of three large companies. (There are also some smaller companies.) In this market, each of these large firms realizes that its own output and the output of each of its competitors *will affect the market price.* In contrast, in a competitive market (like the markets for wheat, corn, or cattle), there are hundreds or thousands of firms supplying the good, and each firm can safely ignore the possible effect of its own output or each competitor's output on the price. In this lesson, we assume that each firm *takes into account* how its own output, and its competitors' outputs, affects the price, and through the price, its own profit.

Since one firm's output decision will affect the profits of the other firms, firms in an oligopoly are likely to act strategically. Two firms are acting *strategically* when each looks at what the other is doing, and thinks along these lines: "I've got to make my production decisions contingent on what he does. If he sells 1,000 units, then to maximize my profits, I have to sell 1,100 units. And if I produce 1,100 units, does it make sense for him to produce 1,000 units?" The firms are *reacting* to each other. In a competitive market, in contrast, the firms do not react to each other;

they only react to the market price, which they take as predetermined or fixed.

In this lesson, we will assume there are only two firms in the market. A market with just two firms is called a *duopoly*. Obviously a duopoly is the simplest sort of oligopoly, and many of the concepts and results that we will describe can be extended to the case of an oligopoly with more than two firms. Duopoly analysis by economists dates back to the 19th century. Some of the central concepts of duopoly analysis have to do with strategic behavior, and the analysis of strategic behavior is the heart of the 20th century discipline called *game theory*. So game theory builds on duopoly theory. We will turn to game theory in the next lesson.

There are two fundamentally different approaches to duopoly theory. The first assumes that duopolists compete with each other through their choices of *quantity*: each firm decides on the quantity it should produce and sell in the market, contingent on the other firm's quantity. The second assumes that duopolists compete with each other through their choices of *price*: each firm decides on the price it should charge, contingent on the price the other firm is charging. The first approach was taken by the French mathematician and economist Antoine Augustin Cournot (1801-1877), who wrote about duopoly in 1838. The second approach was developed by another French mathematician, Joseph Louis Francois Bertrand (1822-1900), in 1883.

We will start in Section 2 by describing the basic Cournot duopoly model, and we will develop that model in Sections 3 and 4. The crucial behavioral assumption of the Cournot model is that each firm assumes the other firm's output is given and fixed, and maximizes its own profit based on that assumption. There are other behavioral assumptions that might be made about the two firms. One is the assumption made by the German economist Heinrich von Stackelberg (1905-1946). Stackelberg assumed, as did Cournot, that the firms make decisions about quantities. But he also assumed, unlike Cournot, that the two firms act differently; one of the duopolists acts as a *follower* (as in Cournot's model), taking the other firm's output as given and fixed, and choosing its own output based on that assumption, but the other duopolist acts as a *leader*, by anticipating that its rival will act as a follower, and choosing its own output based on that knowledge. We will describe the Stackelberg model in Section 5. In Section 6, we will describe the Bertrand model, in which the firms compete with each other through their choices of price, instead of competing, as in Cournot (and Stackelberg) through the choices of quantity. We will see that there are two rather different versions of Bertrand's model, depending on whether the good produced by the two firms is exactly the same (the homogeneous good case), or somewhat different (the differentiated goods case, e.g., Coke and Pepsi).

*2. Cournot Competition*

Assume there are two firms in the market. Firm 1 produces $y_1$ units of the good; firm 2 produces $y_2$ units. The total amount produced is $y = y_1 + y_2$. We assume there is a downward sloping inverse market demand curve $p(y) = p(y_1 + y_2)$. We assume firm $i$ has a cost curve $C_i(y_i)$, for $i = 1, 2$. Firm 1 wants to maximize its profit $\pi_1$, given by:

$$\pi_1(y_1, y_2) = p(y_1 + y_2)y_1 - C_1(y_1).$$

Similarly, firm 2 wants to maximize its profit $\pi_2$, given by:

$$\pi_2(y_1, y_2) = p(y_1 + y_2)y_2 - C_2(y_2).$$

The basic Cournot assumption is this: *when firm 1 chooses its output $y_1$ to maximize its profit, it takes firm 2's output $y_2$ as given and fixed; and, similarly, when firm 2 chooses its output $y_2$ to maximize its profit, it takes firm 1's output $y_1$ as given and fixed.* Therefore when firm 1 differentiates its profit function $\pi_1(y_1, y_2)$, it treats $y_2$ as a constant. This leads to the first order condition

$$\frac{\partial \pi_1}{\partial y_1} = p(y) + \frac{dp(y)}{dy}y_1 - \frac{dC_1(y_1)}{dy_1} = 0.$$

Now firm 1 can solve this equation for $y_1$ as a function of $y_2$. We write the result as

$$y_1 = r_1(y_2).$$

The function $r_1$ is called firm 1's *reaction function.* It shows, for any output level $y_2$ of firm 2, the quantity of the good that firm 1 should produce in order to maximize its profit.

Similarly, firm 2's maximizes its profit subject to the assumption that $y_1$ is a constant. This leads to

$$\frac{\partial \pi_2}{\partial y_2} = p(y) + \frac{dp(y)}{dy}y_2 - \frac{dC_2(y_2)}{dy_2} = 0.$$

Firm 2 can solve this equation for $y_2$ as a function of $y_1$, and we write the result as

$$y_2 = r_2(y_1).$$

The function $r_2$ is firm 2's reaction function. It shows, for any output level $y_1$ of firm 1, the quantity of the good that firm 2 should produce in order to maximize its profit.

Now if the two firms randomly choose their output levels $y_1$ and $y_2$, it is almost certain that neither would be maximizing its profits subject to what the other one is doing. Neither firm would be behaving in a clever way. The result wouldn't make sense. It would be doubly stupid. And if firm 2 randomly chooses an output level $y_2$, and then firm 1 uses its reaction function $r_1$ to choose its output level $y_1$, the result would be half sensible—sensible on the part of firm 1, but stupid on the part of firm 2. But suppose the reaction functions intersect at a point $y_1^*$ and $y_2^*$, and suppose firm 1 chooses $y_1^*$ and firm 2 chooses $y_2^*$. This outcome does make very good sense for both firms, because firm 1 is making the best choice it can, subject to what firm 2 has chosen, and firm 2 is making the best choice it can, subject to what firm 1 has chosen. A *Cournot equilibrium* in a duopoly model is a pair of output levels $y_1^*$ and $y_2^*$ that are consistent in this sense—each firm $i$ is maximizing its profit at $y_i^*$, subject to what the other firm $j$ has chosen, $y_j^*$. The Cournot equilibrium is Augustin Cournot's brilliant solution to the duopoly puzzle.

In short, a Cournot equilibrium is a consistent, self-sustaining, and self-reinforcing outcome in the duopoly model. We now turn to an example to show how the Cournot equilibrium can be found.

**Example 1.** Assume the inverse demand curve is $p(y_1+y_2) = 100-y = 100-y_1-y_2$. Assume the cost curves are $C_1(y_1) = 25y_1$ and $C_2(y_2) = 25y_2$. Marginal cost for either firm is a constant 25. To find firm 1's reaction function, we find the $y_1$ that maximizes $\pi_1(y_1, y_2)$, under the assumption that $y_2$ is constant. Firm 1's profit is:

$$\pi_1(y_1, y_2) = (100 - y_1 - y_2)y_1 - 25y_1.$$

Differentiating with respect to $y_1$ while holding $y_2$ constant, and setting the result equal to zero, gives:

$$\frac{\partial \pi_1}{\partial y_1} = 100 - 2y_1 - y_2 - 25 = 0.$$

Now solving for $y_1$ as a function of $y_2$ gives firm 1's reaction function:

$$y_1 = r_1(y_2) = 37.5 - y_2/2.$$

Firm 2's profit $\pi_2(y_1, y_2)$ is:

$$\pi_2(y_1, y_2) = (100 - y_1 - y_2)y_2 - 25y_2.$$

Differentiating with respect to $y_2$ while holding $y_1$ constant, and setting the result equal to zero, gives:

$$\frac{\partial \pi_2}{\partial y_2} = 100 - 2y_2 - y_1 - 25 = 0.$$

Therefore firm 2's reaction function is:

$$y_2 = r_2(y_1) = 37.5 - y_1/2.$$

In Figure 13.1 below, we show the reaction functions and the Cournot equilibrium in Example 1. The Cournot equilibrium is the point where the two reaction functions intersect. Solving the two reaction function equations simultaneously ($y_1 = 37.5 - y_2/2$ and $y_2 = 37.5 - y_1/2$) easily gives $(y_1^*, y_2^*) = (25, 25)$. At $(25, 25)$, each firm is maximizing its profit, given what the other firm is doing. The market price is $100 - 25 - 25 = 50$. The reader can easily check that profit levels for the firms are $(\pi_1, \pi_2) = (625,\ 625)$. The output levels are mutually consistent; neither firm has an incentive to change, given what the other firm is doing. The Cournot equilibrium $(25, 25)$ makes sense for firm 1, and simultaneously makes sense for firm 2.

Figure 13.1: Draw the two reaction functions on a $(y_1, y_2)$ quadrant. The equilibrium is $y_1^* = y_2^* = 25$.

Caption of Fig. 13.1: The Cournot equilibrium in Example 1.

**Comparison with monopoly and competition.** We can use Example 1 to show how a Cournot equilibrium in a duopoly compares to a monopoly outcome and a competitive outcome. The general result is that in a duopoly (and more generally an oligopoly), total output and price lie somewhere between what they would be under competition or under monopoly.

In Example 1, remember that the Cournot equilibrium price is \$50, and the total quantity is 50. How would we describe the competitive outcome? We would have the same demand curve, but price would be equal to marginal cost. That is, the competitive supply curve would be a horizontal line at $p = 25$. Combining this with the inverse demand curve $p = 100 - y$ gives a competitive equilibrium at $p_C = \$25$ and $y_C = 75$, where the "C" subscript means "competitive."

How would we describe the monopoly outcome? The monopolist would maximize profit by setting marginal revenue equal to marginal cost. In the example, $MR(y) = 100 - 2y$ and $MC(y) = 25$. Therefore $100 - 2y = 25$, or $y = 37.5$. Putting this $y$ in the equation for the inverse demand curve gives $p = 100 - y = 62.5$. In the monopoly solution, then, we have $p_M = \$62.5$ and $y_M = 37.5$, where the "M" subscript stands for "monopoly." We conclude that the Cournot equilibrium in a duopoly lies between the competitive outcome and the monopoly outcome, both for quantity and price.

What about efficiency? We will now investigate the social surplus created at the Cournot equilibrium in the duopoly. As you might expect, the duopoly social surplus lies between the social surplus in the monopoly market, and the social surplus in the competitive market. In short, duopoly (and more generally oligopoly) creates some deadweight loss, but not as much as monopoly creates. We show this in Figure 13.2 below.

The figure is based on Example 1. It shows total output $y$ on the horizontal axis. The outermost line is the inverse demand curve $p(y) = 100 - y$. A monopolist in this market would find the corresponding marginal revenue curve $MR(y) = 100 - 2y$. This is the steeper downward-sloping line shown in the figure. A monopolist would set marginal revenue equal to marginal cost, point $A$ in the figure, to get the quantity $y_M = 37.5$. He would then go up to the demand curve, to point $B$, and get the price $p_M = \$62.5$. Total social surplus under monopoly in this example would be consumers' surplus (the cross-hatched triangle) plus producer's surplus (the cross-hatched square).

If this market were a duopoly, the Cournot equilibrium total quantity would be 50 (shown on the horizontal axis), and the price would be \$50 (shown on the vertical axis). In a transition from a monopoly to duopoly, consumers' surplus would grow and producers' surplus would shrink. But the sum of the two welfare measures would definitely grow. It would grow by the area of the horizontally cross-hatched trapezoid in the figure.

Finally, if this were a competitive market, the equilibrium would require that price equal marginal cost (point $C$ in the figure). In a transition from duopoly to competition, consumers' surplus would greatly expand and producers' surplus would disappear. However, social surplus would definitely grow, by the area of the non-cross-hatched triangle.

Figure 13.2: Based on Example 1, a graph of the demand curve and the corresponding marginal revenue curve. Show $MC(y) = 25$. Show the intercepts on the horizontal and vertical axes. Show consumers' surplus and producer's surplus under monopoly. Show the addition to social surplus from duopoly instead of monopoly. Show the addition to social surplus from competition instead of duopoly.

Caption of Fig. 13.2: Welfare analysis of the duopoly, based on Example 1.

We conclude that the competitive outcome is best for society in the sense that it maximizes social surplus. The Cournot equilibrium in a duopoly is worse than the competitive outcome. The monopoly outcome is the worst of all.

### *3. More on Dynamics*

We have been a little bit vague about how our two firms get to the Cournot equilibrium. The sophisticated and modern game-theory oriented economist looks at Cournot's model and describes it as a *simultaneous move* game. This means that firms 1 and 2, with full knowledge of market demand, and full knowledge of their own cost function and their rival's cost function, choose their output levels, *one time only, and simultaneously.* They end up with a pair of output levels $(y_1, y_2)$. If the pair is a Cournot equilibrium, the outcome makes sense for both firms; it's doubly sensible. If it's not a Cournot equilibrium, the outcome fails to make sense for at least one of the firms, and possibly for both firms. If each firm is a rational profit maximizer and expects the other firm to also be a rational profit maximizer, they should end up at the Cournot equilibrium.

It may be useful to discuss some other possible dynamics in Cournot's model. These descriptions of dynamics necessarily go beyond the simple assumption of simultaneity. One possible dynamic has the firms taking turns reacting to each other. First, firm 1 reacts to firm 2's output; then firm 2 reacts to firm 1's output, and the process goes on until it (hopefully) gets to a Cournot equilibrium. To make this discussion more understandable, let's assume that there is a time dimension, and production and consumption are repeated time unit after time unit, say, day after day.

Let us assume, then, that firms 1 and 2 start at some initial output quantities, on day 0, say, $(y_1^0, y_2^0)$. On the morning of day 1, firm 1 looks at firm 2's output, and calculates what it should produce, contingent on what firm 2 produced on day 0. That is, it goes to its reaction function, and calculates $y_1 = r_1(y_2^0)$. This gives $(y_1^1, y_2^1)$ as the firm outputs on day 1, where $y_1^1 = r_1(y_2^0)$ and where $y_2^1 = y_2^0$.

On the morning of day 2, firm 2 looks at firm 1's output, and calculates what it should produce, contingent on what firm 1 produced on day 1. That is, it goes to its reaction function, and calculates $y_2 = r_2(y_1^1)$. This gives $(y_1^2, y_2^2)$ as the firm outputs on day 2, where $y_1^2 = y_1^1$ and where $y_2^2 = r_2(y_1^1)$.

In other words, the two firms take turns reacting to each other. The process continues, day after day, until it (hopefully) converges to a point where neither firm wants to make further modifications to its daily output. That point is a Cournot equilibrium. Of course, it is slightly odd to think that each firm will use its reaction function at each of its turns, since the reaction functions are based on the assumption that the rival's output is fixed, and that the rival is changing its planned output every other day. (A more rigorous treatment of this and other dynamic adjustment processes is beyond the scope of this book.)

Figure 13.3 illustrates this dynamic story. The process starts at some initial output levels $P^0 = (y_1^0, y_2^0)$, shown on the vertical axis in the figure. On day 1, the process moves to $P^1$; on day 2, it moves to $P^2$, and so on. As in the story of Genesis, on the 7th day they rest. In the figure, the process converges nicely to the Cournot equilibrium. However, this dynamic process would not converge if the reaction functions had the wrong slopes at the equilibrium. The reader is invited to relabel the reaction functions to see what happens if $r_1$ is less steep than $r_2$!

Figure 13.3: As in Figure 13.1, draw the two reaction functions in the $(y_1, y_2)$ quadrant. The equilibrium is at $y_1^* = y_2^* = 25$. Put in a series of horizontal and vertical steps, leading from $P^0$ to $P^1$, and so on, converging at the Cournot equilibrium.

Caption of Fig. 13.3: A dynamic story about the Cournot equilibrium, based on Example 1.

## *4. Collusion*

Let's return to Example 1, and assume that our duopolists are at the Cournot equilibrium. Once again, to make this discussion more understandable, we assume that there is a time dimension, and production and consumption are repeated day after day. Since the two firms are at the Cournot equilibrium, they are producing and selling $y_1^* = y_2^* = 25$, day after day. Given those production levels, the market price is $p = 100 - 25 - 25 = 50$, day after day. Each firm has profits of $\pi_i = py_i^* - 25y_i^* = 625$, day after day.

Now suppose one day the owners of the two firms meet for a game of golf. They have the following conversation: Firm 1 owner: "I'm maximizing my profits at $y_1^* = 25$. But this is based on your holding your output constant at $y_2^* = 25$. What if we both cut output a little bit? Could we make more money that way?" Firm 2 owner: "Well, if we each cut production by one unit, the market price would rise to \$52, since the price is given by $p = 100 - y_1 - y_2$. This means my revenue would change from $\$50 \times 25$ to $\$52 \times 24$. That's almost no change—it's a drop of \$2, to be exact." Firm 1 owner: "But your costs would drop by \$25. So your profit would shoot up." Firm 2 owner: "That's right. In short, if we both cut back output by one unit, your profit would rise by \$23, and mine would too!"

Then their caddy speaks up: "I'm an undercover federal agent. You are both under arrest for colluding and conspiring to fix prices in the market for the gizmos you are producing."

As the presence of our fictional caddy/federal agent suggests, it may be illegal for two duopolists, or more generally a group of firms in an oligopoly, to get together and make plans like this. A *cartel* is a group of producers or firms who organize (or conspire) to raise the price of the good they are selling by restricting supply. Under the antitrust laws of the U.S. and other developed nations, cartels are usually, but not always, illegal. One of the most notorious (but outside the reach of law) cartels of recent history is OPEC, the Organization of Petroleum Exporting Countries. This is an organization of countries whose main purpose is to keep petroleum prices high by controlling production in member countries. Legal cartels in the U.S. include sports leagues, such as Major League Baseball and the National Football League.

What exactly would our two duopolists do if they took it upon themselves to maximize *joint or total profit*, rather than simply letting each firm maximize its own profit, conditional on the other firms's output? As our discussion above suggests, they might gain a lot if they agree to both reduce output.

Let $\pi(y_1 + y_2) = \pi_1(y_1, y_2) + \pi_2(y_1, y_2)$ represent total profit for the two firms combined. Then

$$\pi(y_1, y_2) = p(y_1 + y_2)(y_1 + y_2) - C_1(y_1) - C_2(y_2).$$

The first order conditions for maximizing this function of two variables are:

$$\frac{\partial \pi}{\partial y_1} = p(y_1 + y_2) + \frac{\partial p(y_1 + y_2)}{\partial y_1}(y_1 + y_2) - \frac{dC_1(y_1)}{dy_1} = 0$$

and

$$\frac{\partial \pi}{\partial y_2} = p(y_1 + y_2) + \frac{\partial p(y_1 + y_2)}{\partial y_2}(y_1 + y_2) - \frac{dC_2(y_2)}{dy_2} = 0.$$

Both of these conditions must hold for total profit to be maximized, at least for an interior maximum. (The first order condition for a maximum at a boundary is slightly different.)

In Example 2 below, we will examine joint profit maximization for the simple duopoly introduced in Example 1.

**Example 2.** From Example 1, we have

$$\pi(y_1, y_2) = (100 - y_1 - y_2)(y_1 + y_2) - 25y_1 - 25y_2 = 100y_1 + 100y_2 - y_1^2 - y_2^2 - 2y_1y_2 - 25y_1 - 25y_2.$$

Taking partial derivatives with respect to $y_1$ and $y_2$, we get

$$\frac{\partial \pi}{\partial y_1} = 100 - 2y_1 - 2y_2 - 25 = 0$$

or

$$y_1 + y_2 = 37.5.$$

Similarly,

$$\frac{\partial \pi}{\partial y_2} = 100 - 2y_1 - 2y_2 - 25 = 0$$

or

$$y_1 + y_2 = 37.5.$$

The first order conditions for joint profit maximization are identical, because the two firms have identical cost curves. We conclude that joint profit maximization requires $y_1 + y_2 = 37.5$. For example, each firm could produce $y_i = 37.5/2 = 18.75$. With these levels of output, $\pi_1(y_1, y_2) = \pi_2(y_1, y_2) = (100 - 37.5)18.75 - (25)18.75 = 703.125$. Each firm would be making \$703.125. This is considerably better than the \$625 profit for each firm at the Cournot equilibrium.

In Figure 13.4 below, we show the joint profit maximization points, the collusion outcomes, for this duopoly example. The shaded line is the set of outcomes which maximize joint profits, that is, the set for which $y_1 + y_2 = 37.5$. Note that $(18.75, 18.75)$ is one of many possibilities, but they all involve total output of 37.5 units. Finally, the reader should remember our Figure 13.2 comparison of monopoly, duopoly, and competition. In conjunction with that figure, we determined that a monopoly firm would produce 37.5 units. In Example 2 and Figure 13.4, the two duopolists together are producing a total of 37.5 units. We get the same answer because the duopolists in Example 2 are acting just like a monopolist!

Figure 13.4: Similar to Figure 13.1, but include the set of collusion outcomes.

Caption of Fig. 13.4: The reaction curves, the Cournot equilibrium, and the collusion outcomes, all based on Example 1.

Happily for consumers of their products, cartels and colluding duopolists are inherently unstable. Because a collusion agreement is not a Cournot equilibrium, each firm has an incentive to cheat on the agreement. For instance, to continue our numerical example, suppose the two duopolists have agreed to be at the joint profit maximizing point $(18.75, 18.75)$. Some time later, the owner of firm 1 wakes up one morning, and says to himself: "The hell with that lawbreaking S.O.B. If he's going to produce 18.75 units per day, I shall greatly increase my own profits by using my reaction function to figure out what I should produce." The answer is

$$y_1 = r_1(y_2) = 37.5 - y_2/2 = 37.5 - 18.75/2 = 28.125.$$

As soon as the owner of firm 1 figures this out, he produces 28.125 units per day. This raises firm 1's profits from \$703.125 per day to $\pi_1(y_1, y_2) = (100 - 28.125 - 18.75)28.125 - 25(28.125) = \$791$ per day, a gain of nearly \$88. Shortly thereafter, the owner of Firm 2 realizes he's been duped. So firm 2 reacts to firm 1's output of $y_1 = 28.125$. Firm 2 switches to

$$y_2 = r_2(y_1) = 37.5 - y_1/2 = 37.5 - 28.125/2 = 23.44.$$

And so it goes. After a few rounds of this reacting and re-reacting, the duopoly may end up back at, or near, the Cournot equilibrium of $(25, 25)$.

The point of this discussion is that duopolists, and more generally members of cartels, always have incentives to collude, to get together and plot against the public, to figure out how they might reduce output and increase their joint profits. However, having come to some kind of collusion agreement, the duopolists, or the cartel members, will be tempted to cheat. If they do start to cheat, they are likely to drift back toward a Cournot equilibrium. This, then, is the big dynamic: independent profit maximization leads toward the Cournot solution. Then joint profit maximization leads toward the collusion solution. Unless the firms can enforce their collusion agreements, cheating and independent profit maximization leads back toward the Cournot solution. So turns the world of duopoly, or the world of cartels. In the absence of collusion enforcement mechanisms, the likely prediction for a duopoly, or for a cartel, is instability.

### *5. Stackelberg Competition*

We will now turn to the duopoly model of the German economist Heinrich von Stackelberg. Stackelberg assumed that one of the duopoly firms acts like a Cournot duopolist. That is, it takes the other firm's output as given and fixed, and it chooses its own output based on that assumption. We call this firm the *follower*. Stackelberg assumed that the other firm anticipates this behavior, and maximizes its profit based on the assumption that its rival is a follower. We call this firm the *leader*.

Recall that in the analysis of the Cournot equilibrium, we formally assumed a simultaneous move structure. That is, we assumed the interaction between the two firms was *one time only, and simultaneous.* Our extensive informal discussion of dynamics involved stories about day-by-day interactions, reactions, and re-reactions, but that discussion was not necessary for the formal definition of the Cournot equilibrium.

In order to describe the Stackelberg model, we now formally assume that the interaction between the two firms is *in two steps, sequentially.* In the first step, the leader firm determines its planned output. In the second and final step, the follower firm determines its output. The firms then produce and sell their outputs, at a market price contingent on $y_1 + y_2$, and make their profits.

Here is how it works. We will let firm 1 be the leader firm; its output is $y_1$. Firm 2 is the follower firm; its output is $y_2$. The follower firm is the second firm to act. It knows what the leader firm is producing, because that was determined (and announced) at step one. The follower firm acts just like a firm in the Cournot analysis, it takes $y_1$ as given, and determines what maximizes its own profits given $y_1$. Therefore it uses its reaction function to determine its output. That is, $y_2 = r_2(y_1)$.

But firm 1, the leader firm, knows how firm 2, the follower firm, behaves. That is, firm 1 anticipates that firm 2 will choose $y_2$ by using its reaction function formula. Therefore firm 1 can use its knowledge of firm 2's behavior, and use the fact that it goes first and firm 2 goes second. It does this in a simple way; it just substitutes $r_2(y_1)$ for $y_2$ in its own profit function. This gets rid of the $y_2$ terms; firm 1's profit is now simply a function of $y_1$, and firm 1 simply chooses $y_1$ to maximize profits.

Formally, firm 1's profit is $\pi_1(y_1, y_2) = p(y_1 + y_2)y_1 - C_1(y_1)$. Substituting $r_2(y_1)$ for $y_2$ makes this a function of one variable only:

$$\pi_1(y_1) = p(y_1 + r_2(y_1))y_1 - C_1(y_1).$$

The first order condition for profit maximization is to set the derivative of profit with respect to $y_1$ equal to zero. This gives:

$$\frac{d\pi_1(y_1)}{dy_1} = p(y_1 + r_2(y_1)) + y_1\frac{dp}{dy}(1 + \frac{dr_2(y_1)}{dy_1}) - MC_1(y_1) = 0.$$

Let's apply the result to Example 1. Recall that in that example, $p(y) = 100 - y = 100 - y_1 - y_2$, $C_1(y_1) = 25y_1$, and $r_2(y_1) = 37.5 - y_1/2$. Therefore $\frac{dp}{dy} = -1$, and $\frac{dr_2(y_1)}{dy_1} = -1/2$. Plugging into the first order condition then gives:

$$(100 - y_1 - (37.5 - y_1/2)) - y_1(1 - 1/2) - 25 = 0.$$

This gives $y_1 = 37.5$. Putting this into the follower's reaction function gives:

$$y_2 = r_2(y_1) = 37.5 - 37.5/2 = 18.75.$$

Note that we know that a Stackelberg leader firm will never end up with a profit level lower than what it gets at the Cournot equilibrium. One option always available to the leader is to announce its Cournot output, to which the follower would respond with its Cournot output. This would produce the Cournot profits for the two firms.

Figure 13.5 below shows the Cournot equilibrium, the collusion outcomes, and the Stackelberg equilibrium for the numerical example we have been using throughout this lesson.

Figure 13.5: Similar to Figure 13.4, but now add the Stackelberg equilibrium, when firm 1 is the leader and firm 2 is the follower.

Caption of Fig. 13.5: The reaction curves, the Cournot equilibrium, the collusion outcomes, and the Stackelberg equilibrium, all based on Example 1.

### *6. Bertrand Competition*

We will now turn to the model developed by the mathematician Joseph Louis Francois Bertrand (1822-1900). While the Cournot model of duopoly assumes that each of the two firms decides on what *quantity to produce*, the Bertrand model assumes that each firm decides on what *price to charge*. As we shall see, this approach can lead to a very interesting but somewhat unrealistic model, with implications that are very different than the implications of the Cournot model. Or it can lead to a model that is perhaps more realistic than Cournot's, but with implications similar to Cournot's. This difference arises because we can develop the Bertrand analysis in either of two ways:

(1) We can assume, as we assumed for the Cournot model, that the two firms are producing *exactly the same good*. That is, whether produced by firm 1 or firm 2, a unit of the good is a unit of the good, as far as the buyers are concerned. This is the property of *homogeneity*. Commodities like electricity, oil, metals, or wheat, are *homogeneous*; whether produced by firm 1 or firm 2, a gallon of fuel oil is a gallon of fuel oil. For a homogeneous good being produced and sold by two firms, there can be only one price; if firm 1 tries to sell it at a slightly higher price than firm 2, its sales drop to zero.

(2) Alternatively, we can assume that the two firms produce goods which are similar but slightly different, or *differentiated*. Think of Coke and Pepsi, McDonald's and Burger King, Bud Light and Miller Lite, or Schick and Gillette. If the two firms produce goods which are differentiated, they can charge different prices, and in fact commonly do so. Each is likely to claim that its good is both better *and* less expensive.

**Homogeneous goods case.** Let us now assume that firms 1 and 2 produce exactly the same good. For simplicity, we will assume that the two firms have identical constant marginal cost functions. Let $MC$ represent the marginal cost of producing a unit of the good, and let $MC$ be the same for both firms and constant over all output levels. Let $y_1$ and $y_2$ represent the production levels of the two firms. We assume the firms set prices $p_1$ and $p_2$, and then sell their output to meet demand. Since the good is homogeneous, if firm $i$ sets a lower price than firm $j$, then all the customers will buy from firm $i$. This implies that in any equilibrium where both firms are operating, we must have $p_1 = p_2 = p$, where $p$ is the (single) market price. If there is one price $p$, the market demand curve is given by a function $y = y(p)$. We will assume that if both firms are charging the same price $p$, they will split the market demand equally—each will sell $y_1 = y_2 = y(p)/2$ units of the good.

This model cannot be solved using standard calculus techniques. This is because although the function $y(p)$ is well behaved, firm $i$'s demand function $y_i(p_i)$ is not. It has a sharp discontinuity when $p_i$ equals $p_j$. If $p_i < p_j$, demand for firm $i$ is $y(p_i)$; if $p_i = p_j$, demand for firm $i$ is $y(p_i)/2$; and if $p_i > p_j$, demand for firm $i$ is zero.

Since we cannot use standard calculus techniques, we must reason along more abstract lines. Recall our definition of a Cournot equilibrium from Section 2 above. In a model where the two duopolists are reacting to each other by setting quantities, a Cournot equilibrium is a pair of output levels $y_1^*$ and $y_2^*$ that are consistent, in the sense that each firm $i$ is maximizing its profit at $y_i^*$, subject to what the other firm $j$ has chosen, $y_j^*$. Let us now define an equilibrium in a similar way, but for the current model where the two duopolists are reacting to each other by setting prices. A *Bertrand equilibrium* is a pair of prices $p_1^*$ and $p_2^*$ that are consistent, in the sense that each firm $i$ is maximizing its profit with the choice of $p_i^*$, subject to what the other firm $j$ has chosen, $p_j^*$.

What can we say about a Bertrand equilibrium in the homogeneous goods case? Let $(p_1^*, p_2^*)$ represent the equilibrium prices and let $(y_1^*, y_2^*)$ the corresponding equilibrium quantities. Here's what we can conclude:

(1) The firms must be charging the same price. That is, $p_1^* = p_2^* = p^*$. Suppose to the contrary that they are charging different prices, and without loss of generality, assume $p_1^* < p_2^*$. Then firm 1 is selling a positive quantity of the good, and firm 2 is selling nothing.

(a) If $p_1^* < MC$, then firm 1 has negative profits and would be better off shutting down. So this cannot be an equilibrium.

(b) If $p_1^* = MC$, firm 1 is making \$0 on each unit it produces and sells. It could increase its price somewhat, while keeping it below $p_2^*$, and make positive amounts on all the units it sells. (It would sell fewer units, but it would make money on each one.) So this cannot be an equilibrium.

(c) If $p_1^* > MC$, firm 1 is making positive profits on all the units it produces and sells. But if this were the case, firm 2 would gain by entering the market with a price strictly between $MC$ and $p_1^*$, taking all of firm 1's customers away. So this cannot be an equilibrium either.

We have established that $p_1^* \neq p_2^*$ implies we cannot have a Bertrand equilibrium. Therefore, at a Bertrand equilibrium, we must have $p_1^* = p_2^* = p^*$. Since we have assumed that demand is split equally between the two firms when their prices are the same, therefore $y_1^* = y_2^* = y(p^*)/2$.

(2) Marginal cost cannot be less than price; that is, $MC < p^*$ cannot hold. Here's why. Suppose the inequality held. Assume for concreteness that $MC = 25$ and that $p_1^* = p_2^* = p^* = 26$. Then either firm, say, firm 1, could shave its price to $p_1 = 25.99$. By doing so, it would steal away all of firm 2's customers (half of the total market), make almost a dollar profit on each of those sales, while giving up a penny's profit on each of the sales it already had (half of the total market). Its profits would obviously go way up. This contradicts our assumption that firm 1 is choosing a price that maximizes its profit subject to what firm 2 has chosen.

(3) Marginal cost cannot be greater than price; that is, $MC > p^*$ cannot hold. With constant marginal costs, for either firm $i$, $MC > p^* = p_i^*$ would imply negative profits, and the firm would opt to go out of business, rather than sell $y_i^*$ at a price of $p^*$.

(4) For both firms, marginal cost equals price; that is, $MC = p^* = p_1^* = p_2^*$. This obviously follows from (2) and (3).

We conclude that in a Bertrand equilibrium, in the homogeneous good case, under the assumptions we have made, *firms 1 and 2 will charge the same price, and the price will be equal to marginal cost. But this means that the duopoly market, in the Bertrand model with a homogeneous good, looks just like a competitive market. In particular, there is no inefficiency (no loss of social surplus) in the duopoly market.*

**Differentiated goods case.** Now we assume that firms 1 and 2 produce goods that are differentiated—similar, but not identical. (Think of McDonald's and Burger King.) We continue to let $y_1$ and $y_2$ represent the outputs of the two firms, and $p_1$ and $p_2$ represent the prices. Since the goods are different, $p_i < p_j$ *does not* imply that firm $j$'s sales will drop to zero, and the firms *will not be forced* to charge the same price in equilibrium.

Now there are demand functions for each of the two firms that depend on the two prices: $y_1 = y_1(p_1, p_2)$ and $y_2 = y_2(p_1, p_2)$. For firm $i$'s demand function $y_i$, the partial derivative with respect to $p_i$ is assumed to be negative (as $i$ raises its price, demand for $i$'s good falls), but the partial derivative with respect to $p_j$ is positive (as $i$'s competitor $j$ raises its price, demand for $i$'s good rises).

Firm 1's profit is

$$\pi_1(p_1, p_2) = p_1 y_1(p_1, p_2) - C_1(y_1(p_1, p_2)),$$

and firm 2's profit is

$$\pi_2(p_1, p_2) = p_2 y_2(p_1, p_2) - C_2(y_2(p_1, p_2).$$

We write these as functions of the two prices $(p_1, p_2)$ because each firm chooses its own price, rather than its own quantity as in the Cournot model. Firm 1 chooses $p_1$ to maximize $\pi_1(p_1, p_2)$, taking $p_2$ as given and fixed; firm 2 chooses $p_2$ to maximize $\pi_2(p_1, p_2)$, taking $p_1$ as given and fixed.

Firm 1's first order condition is

$$\frac{\partial \pi_1}{\partial p_1} = y_1(p_1, p_2) + \frac{\partial y_1(p_1, p_2)}{\partial p_1} p_1 - \frac{dC_1(y_1)}{dy_1} \frac{\partial y_1(p_1, p_2)}{\partial p_1} = 0.$$

We can write this more compactly as

$$\frac{\partial \pi_1}{\partial p_1} = y_1 + \frac{\partial y_1}{\partial p_1} p_1 - MC_1 \frac{\partial y_1}{\partial p_1} = 0.$$

Similarly, firm 2's first order condition is

$$\frac{\partial \pi_2}{\partial p_2} = y_2 + \frac{\partial y_2}{\partial p_2} p_2 - MC_2 \frac{\partial y_2}{\partial p_2} = 0.$$

Recall that when we analyzed the Cournot model, we used the two firms' first order conditions to derive reaction functions. We can do the same here, using firm $i$'s first order condition to find a reaction function that shows firm $i$'s profit-maximizing price as a function of firm $j$'s price: $p_i = r_i(p_j)$.

To find a Bertrand equilibrium, we look for a pair of prices $p_1^*$ and $p_2^*$ that are consistent in the sense that each firm $i$ is maximizing its profit with the choice of $p_i^*$, subject to what the other firm $j$ has chosen, $p_j^*$. We will again let $y_1^*$ and $y_2^*$ represent the corresponding equilibrium quantities. In the differentiated goods case, we use the first order conditions for profit maximization (or the reaction functions) to find the equilibrium. We will illustrate with an example below. Before we do, however, let's use firm 1's reaction function to show that in the differentiated goods Bertrand model (unlike the homogeneous good Bertrand model), at equilibrium, the price will be *greater than marginal cost.*

Here's why. At the equilibrium, firm 1's first order condition must be satisfied, which gives:

$$y_1^* + \frac{\partial y_1}{\partial p_1} p_1^* - MC_1(y_1^*) \frac{\partial y_1}{\partial p_1} = 0.$$

Rearranging gives:

$$y_1^* = -\frac{\partial y_1}{\partial p_1}(p_1^* - MC_1(y_1^*)).$$

But $y_1^*$ is *positive*, $\frac{\partial y_1}{\partial p_1}$ is *negative*, and therefore $p_1^* - MC_1(y_1^*)$ is *positive*, or

$$p_1^* > MC_1(y_1^*).$$

That is, in the differentiated goods case, at the Bertrand equilibrium, *price is greater than marginal cost* for firm 1, and similarly for firm 2. In short, *in the differentiated goods case, a Bertrand equilibrium has social welfare properties similar to the Cournot equilibrium discussed in Section 2 above.*

**Example 3.** We again assume the cost curves are $C_1(y_1) = 25y_1$ and $C_2(y_2) = 25y_2$, and so $MC_1 = MC_2 = 25$. We now assume the demand functions for firms 1 and 2 are

$$y_1(p_1, p_2) = 50 - p_1 + p_2/2$$

and

$$y_2(p_1, p_2) = 50 - p_2 + p_1/2.$$

Firm 1 wants to choose $p_1$ to maximize

$$\pi_1(p_1, p_2) = p_1(50 - p_1 + p_2/2) - 25(50 - p_1 + p_2/2),$$

taking $p_2$ as given. The first order condition gives firm 1's reaction function,

$$p_1 = r_1(p_2) = 75/2 + p_2/4.$$

Similarly, firm 2 wants to choose $p_2$ to maximize

$$\pi_2(p_1, p_2) = p_2(50 - p_2 + p_1/2) - 25(50 - p_2 + p_1/2).$$

The first order condition leads to firm 2's reaction function,

$$p_2 = r_2(p_1) = 75/2 + p_1/4.$$

Solving the two reaction functions (or the two first order conditions) simultaneously gives the Bertrand equilibrium prices of $p_1^* = 50$ and $p_2^* = 50$. The equilibrium quantities are $y_1^* = 25$ and $y_1^* = 25$. Equilibrium profit levels are easily calculated. For firm 1,

$$\pi(p_1^*, p_2^*) = p_1^* y_1^* - C_1(y_1^*) = 50 \times 25 - 25 \times 25 = 625,$$

and similarly for firm 2. Figure 13.6 below shows the reaction functions and the equilibrium prices for Example 3.

Figure 13.6: Draw the two reaction functions on a $(p_1, p_2)$ quadrant. Their intercepts are at 37.5 and 37.5 on the horizontal and vertical axes. They intersect at the equilibrium $p_1^* = p_2^* = 50$.

Caption of Fig. 13.6: The Bertrand equilibrium in Example 3.

## *7. A Solved Problem*

**The Problem**

Recall the inverse demand function assumed in Example 1:

$$p(y_1 + y_2) = 100 - y_1 - y_2.$$

The cost functions in that example were $C_1(y_1) = 25y_1$ and $C_2(y_2) = 25y_2$. In order to find the Cournot equilibrium, we used the reaction functions of firms 1 and 2:

$$y_1 = r_1(y_2) = 37.5 - y_2/2, \qquad \text{and} \qquad y_2 = r_2(y_1) = 37.5 - y_1/2.$$

Recall from Example 1 that the Cournot equilibrium was $(y_1^*, y_2^*) = (25, 25)$, and the profit levels at the Cournot equilibrium were $(\pi_1, \pi_2) = (625, 625)$.

(a) Now assume firm 1 is a Stackelberg leader, and firm 2 is a Stackelberg follower. Calculate the Stackelberg equilibrium price and quantities. Also find the Stackelberg equilibrium profit levels.

(b) What would happen if both firms believed that they were Stackelberg leaders?

**The Solution**

(a) If firm 1 is a Stackelberg leader and firm 2 is a Stackelberg follower, firm 2 acts like a standard Cournot firm; it takes firm 1's output $y_1$ as given and fixed and chooses its own output in response. In other words, it chooses $y_2$ to maximize $\pi_2(y_1, y_2)$, under the assumption that $y_1$ is constant. This means that it derives and uses its reaction function $y_2 = r_2(y_1) = 37.5 - y_1/2$. Firm 1, on the other hand, knows that $y_2$ is not fixed; in fact, firm 1 knows exactly how firm 2 chooses $y_2$ based on $y_1$. In other words, firm 1 knows firm 2's reaction function and exploits that knowledge.

We can now write firm 1's profit function as

$$\pi_1(y_1, y_2) = p(y_1 + y_2)y_1 - C_1(y_1) = p(y_1 + r_2(y_1))y_1 - C_1(y_1) = (100 - y_1 - r_2(y_1))y_1 - 25y_1.$$

The next step is to substitute for $r_2(y_1)$, using $r_2(y_1) = 37.5 - y_1/2$. After a little minor algebra and some rearranging, we get $\pi_1(y_1) = 37.5y_1 - y_1^2/2$. We differentiate this function, set the result equal to zero and solve, which produces $y_1^* = 37.5$. The $^*$ denotes the Stackelberg equilibrium quantity for firm 1, the leader firm. Plugging in $y_1^*$ into firm 2's reaction function will allow us to solve for the follower firm's Stackelberg equilibrium quantity: $y_2^* = 37.5 - y_1^*/2 = 18.75$.

To find the Stackelberg equilibrium price, we insert $y_1^*$ and $y_2^*$ in the inverse demand function, which gives $p^* = 100 - 37.5 - 18.75 = 43.75$. Profit levels are given by $\pi_1^* = p^* y_1^* - 25y_1^* = (p^* - 25)y_1^* = (43.75 - 25)37.5 = 703.13$. Similarly, $\pi_2^* = (p^* - 25)y_2^* = (43.75 - 25)18.75 = 351.56$.

(b) If firm 1 is a Stackelberg leader, it knows firm 2's reaction function and takes advantage of it. In part (a), we found that this line of reasoning would lead firm 1 to choose $y_1^* = 37.5$. Let us now assume that firm 2 also acts like a Stackelberg leader. It figures out firm 1's reaction function $y_1 = r_1(y_2) = 37.5 - y_2/2$, plugs this into its own profit function $\pi_2 = (100 - r_1(y_2) - p_2)y_2 - C_2(y_2)$, and maximizes, which leads to $y_2^* = 37.5$. Now the market price would be $p^* = 100 - 37.5 - 37.5 = 25$ and profit levels for each firm is $\pi_i^* = (25 - 25)37.5 = 0$. In their quest for leadership, they end up with nothing.

This situation, however, is quite peculiar in that firm 1 believes firm 2 is choosing its output according to its reaction function, and firm 2 believes firm 1 is choosing its output according to its reaction function. But both beliefs are false. (We could call this situation a Stackelberg "disequilibrium.")

*Exercises*

1. (From H 9.1) The Corleone and Chung families are the only providers of good $h$ in the U.S. The market demand for good $h$ is $h = 1,200 - 20p$. The costs of production for each of them are represented by the cost functions $C_1(h_1) = 10h_1$ and $C_2(h_2) = 20h_2$, respectively. Suppose both families must choose their output levels simultaneously.

   (a) Derive their reaction functions.

   (b) Calculate the Cournot equilibrium in this market. Indicate output levels, market price, and individual profits.

2. (From H 9.1) Consider the Corleone and Chung families from Question 1. Suppose the two families sign an agreement to restrict the amount of good $h$ in the market. By doing this, market price and profits will increase. Suppose the agreement specifies that given the cost differential, Corleone will receive 3/5 of the total profits and Chung will receive 2/5.

   (a) Find the solution to this collusion problem. Indicate individual outputs, market price, total profits, and individual profits.

   (b) Does either family have an incentive to break the agreement? Who and why?

   Hint: Remember that the first order conditions obtained from differentiation gives an interior solution. If the first order conditions do not yield a solution, try cases in which one of the firms produces zero.

3. (From H 9.2) *MBI* and *Pear* are the only two producers of computers. *MBI* started producing computers earlier than *Pear*. *MBI*'s costs of production are given by $C_1(y_1) = y_1^2$. *Pear*'s cost function is $C_2(y_2) = 5y_2$. The national demand for computers is $y = 10^6 - 10^5 p$.

   (a) Calculate the Stackelberg equilibrium in which *MBI* is the leader in this market. Indicate output levels, market price, and the profits of each firm.

   (b) Suppose that both firms enter this market at the same time. Calculate the Cournot equilibrium and compare it to the situation in part (a).

4. Reuben and Simeon are duopolists producing jeans in a differentiated goods market. The market demand for Reuben's jeans is $y_1 = 80 - p_1 + \frac{1}{2}p_2$, while the market demand for Simeon's jeans is $y_2 = 160 - p_2 + \frac{1}{2}p_1$. Reuben's cost function is $C_1(y_1) = 80y_1$ while Simeon's cost function is $C_2(y_2) = 160y_2$.

   (a) Calculate the Bertrand equilibrium in this market. Indicate each firm's price, output level, and profits.

   (b) Find prices and output levels that would maximize joint profits, and calculate the maximum joint profits.

5. Laban and Jacob are sheep farmers in a differentiated goods market. The market demand for Laban's sheep wool is $y_1 = 34 - p_1 + \frac{1}{3}p_2$, while the market demand for Jacob's sheep wool is $y_2 = 40 - p_2 + \frac{1}{2}p_1$. Laban's cost function is $C_1(y_1) = 24y_1$, while Jacob's cost function is $C_2(y_2) = 20y_2$. They compete with each other through their choices of price.

   (a) Calculate the equilibrium in which Laban is the price leader in this market, and Jacob is the price follower. Indicate prices, output levels, and individual profits.

   (b) How do prices, output levels, and individual profits change if Jacob is the price leader in this market, and Laban is the price follower?

6. Compare the social welfare properties of the following models of duopoly behavior: simultaneous quantity setting (Cournot), quantity leadership (Stackelberg), simultaneous price setting (Bertrand, both homogeneous goods and differentiated goods cases), price leadership, and collusion. Which model results in the highest output? The lowest output? The highest price? The lowest price?

# Index

Bertrand competition, 14

Bertrand duopoly model, 2

Bertrand equilibrium, 15

Bertrand, J., 2, 14

cartel, 9

collusion, 8, 11

competitive equilibrium, 5, 7

Cournot competition, 3

Cournot equilibrium, 4, 5, 7

Cournot, A., 2

deadweight loss, 6

differentiated goods, 17, 18

duopoly, 2

game theory, 2

homogeneity, 14

homogeneous goods, 14, 15

monopoly, 5–7

oligopoly, 1

Stackelberg competition, 12

Stackelberg duopoly model, 2

Stackelberg follower, 2, 12

Stackelberg leader, 2, 12

Stackelberg, H., 2, 12

strategic behavior, 1, 2

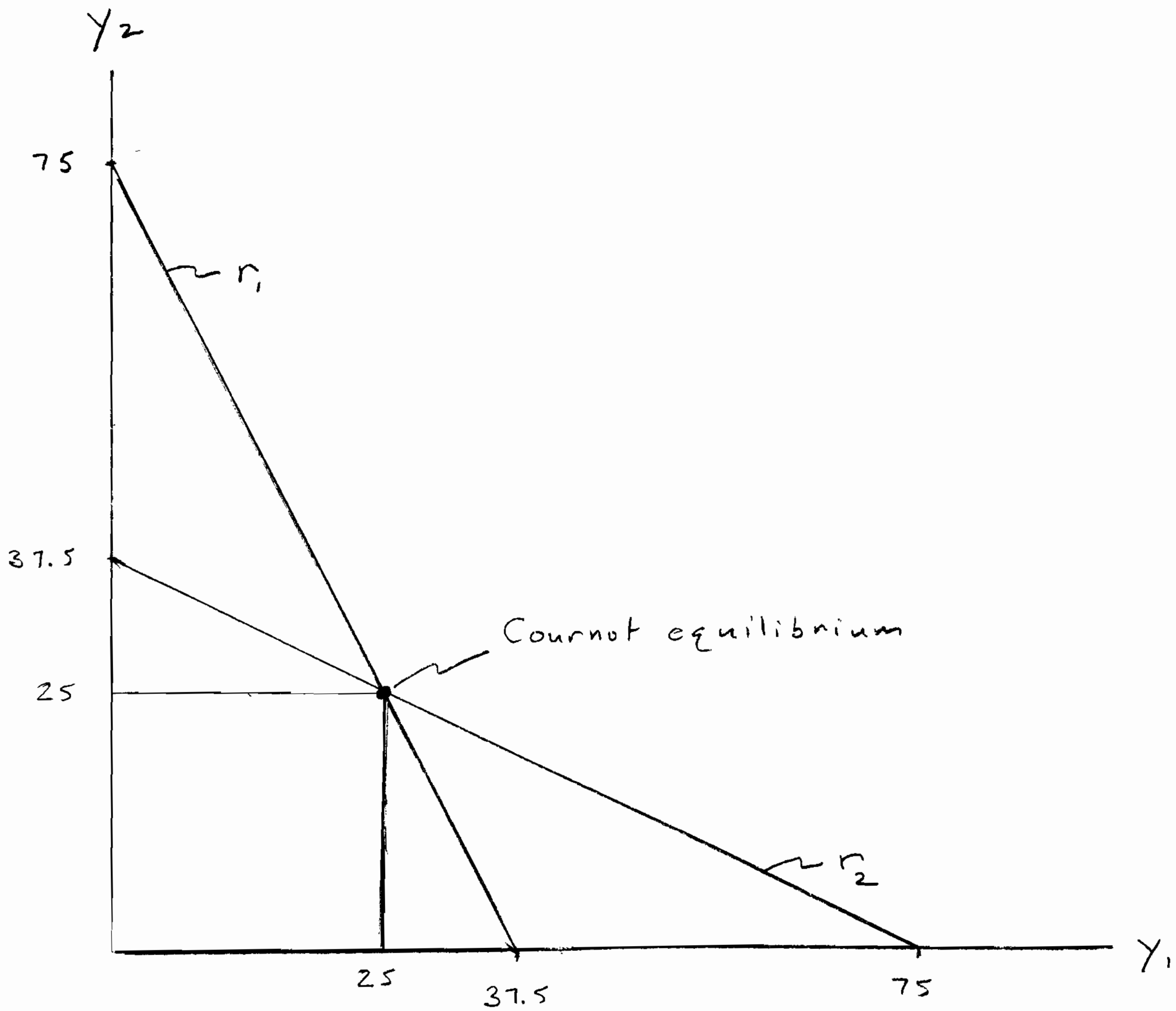



Figure 13.1

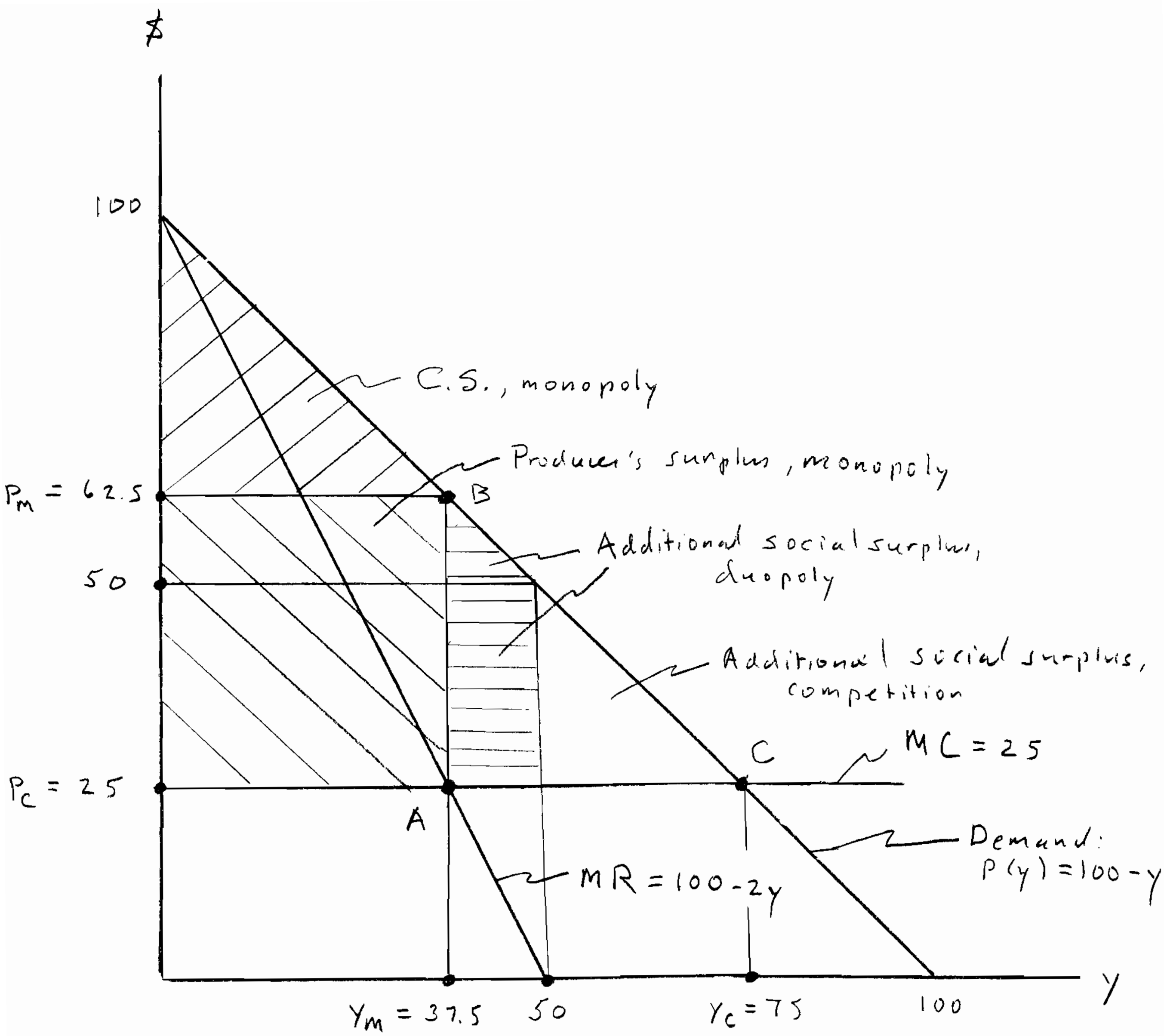

$
100
C.S., monopoly
Producer's surplus, monopoly
$P_m$ = 62.5
B
Additional social surplus, duopoly
50
Additional social surplus, competition
MC = 25
C
$P_c$ = 25
A
Demand: P(y) = 100 - y
MR = 100 - 2y
$Y_m$ = 37.5
50
$Y_c$ = 75
100
y

Figure 13.2

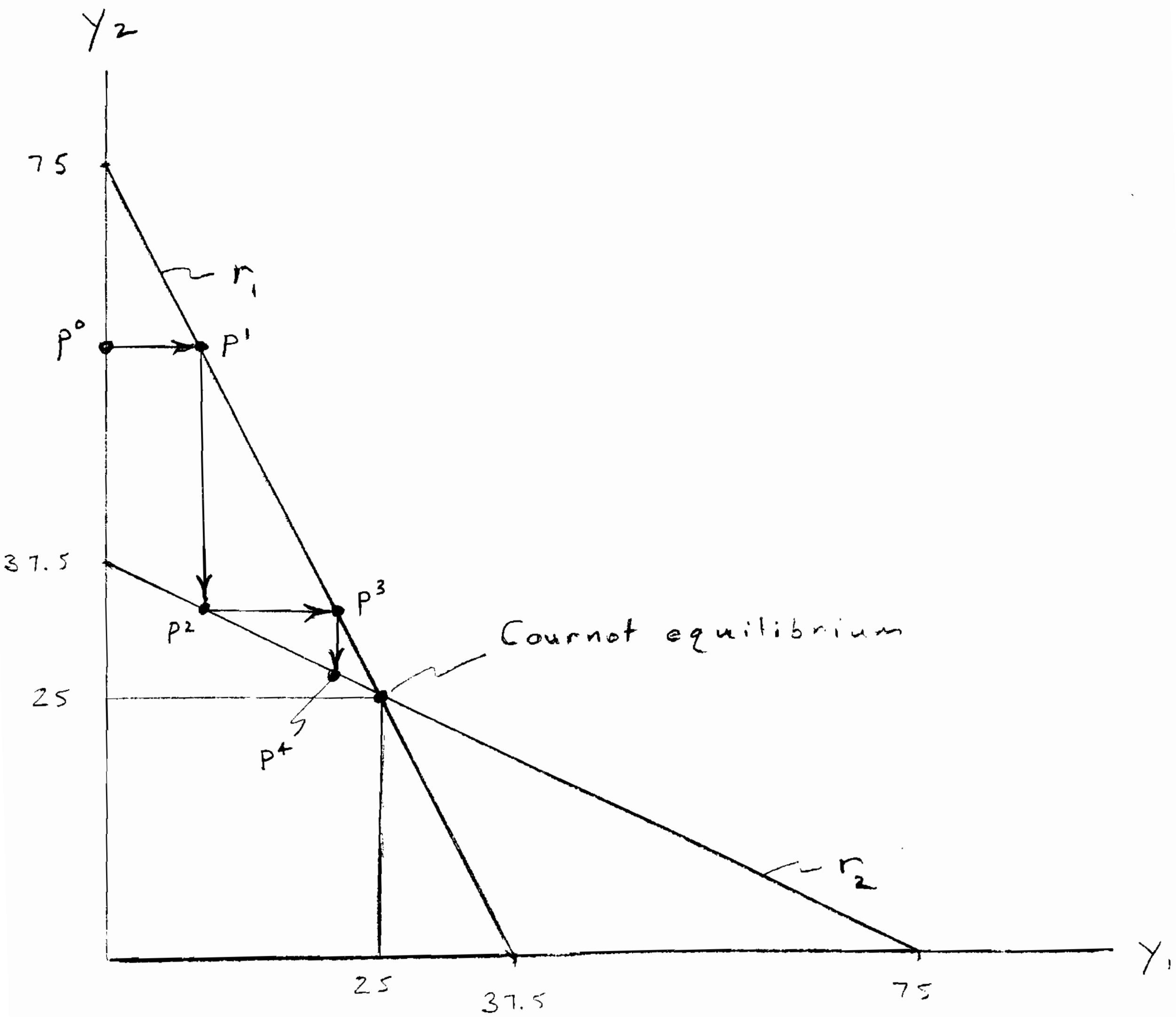
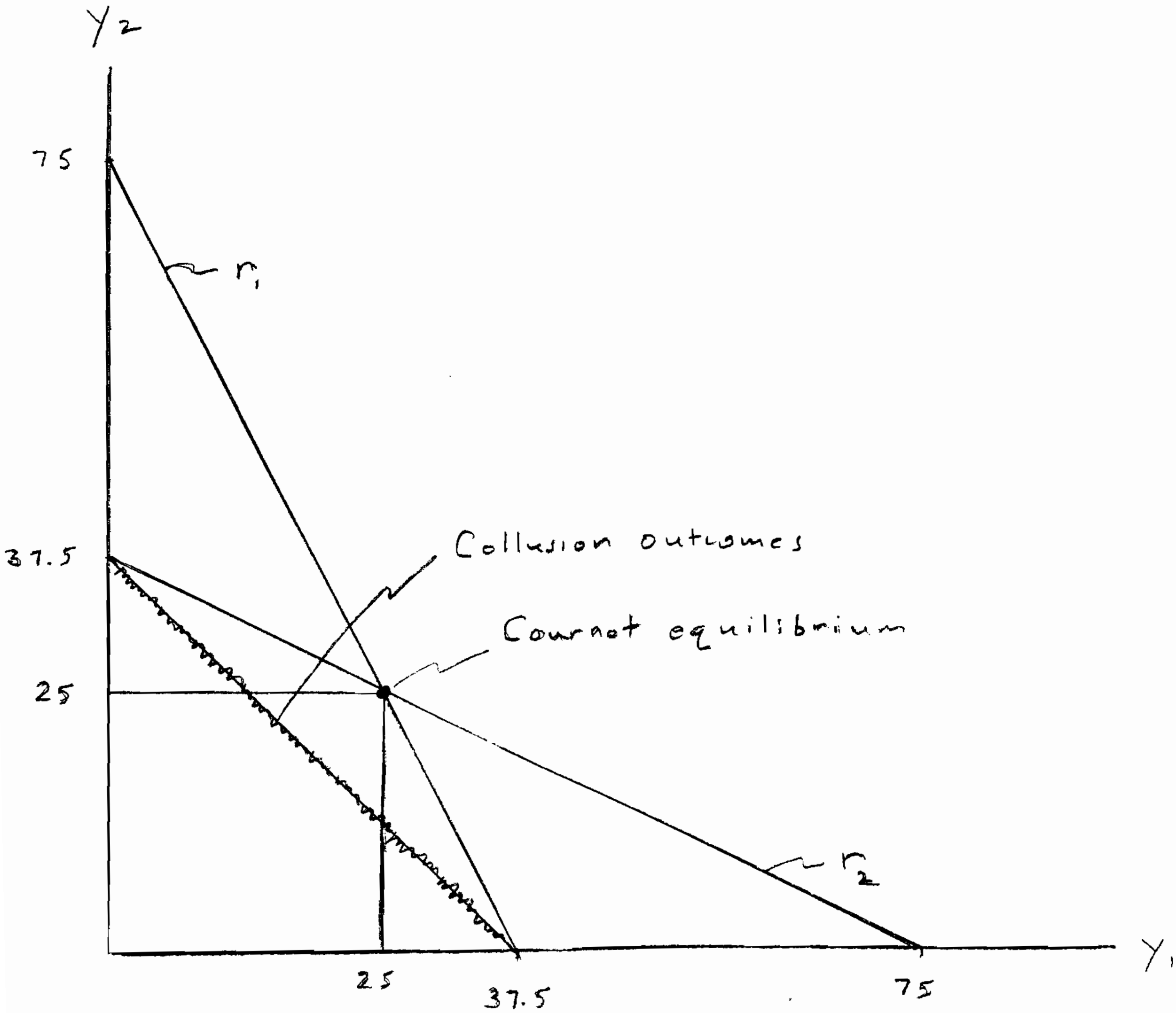
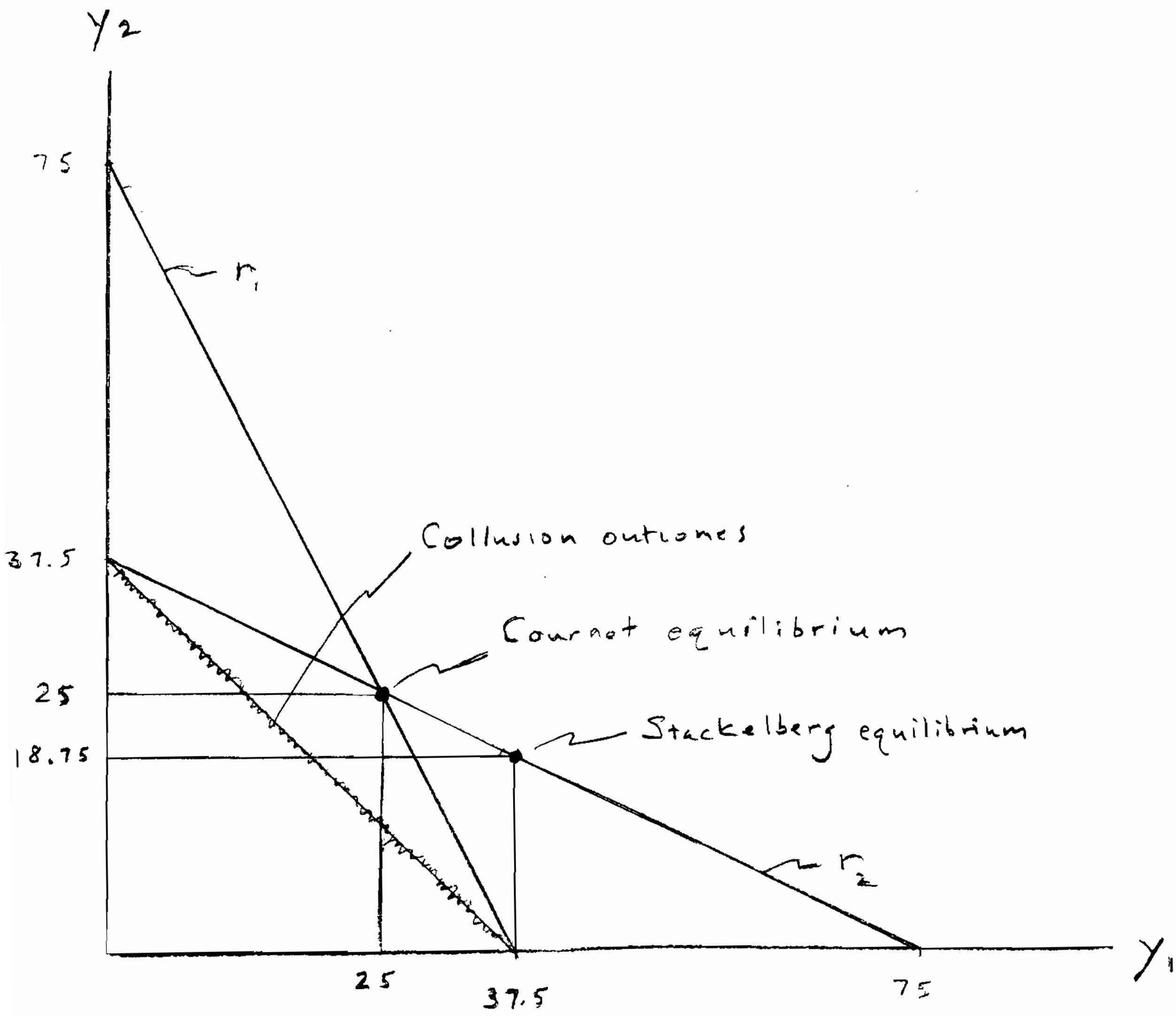

Y2
75
r1
P0
P1
37.5
P3
P2
Cournot equilibrium
25
P4
r2
Y1
25
37.5
75

Figure 13.3



Y2
75
37.5
25
r1
Collusion outcomes
Cournot equilibrium
r2
Y1
25
37.5
75


Figure 13.4



$y_2$
75
$r_1$
37.5
Collusion outcomes
Cournot equilibrium
25
Stackelberg equilibrium
18.75
$r_2$
25
37.5
75
$y_1$


Figure 13.5

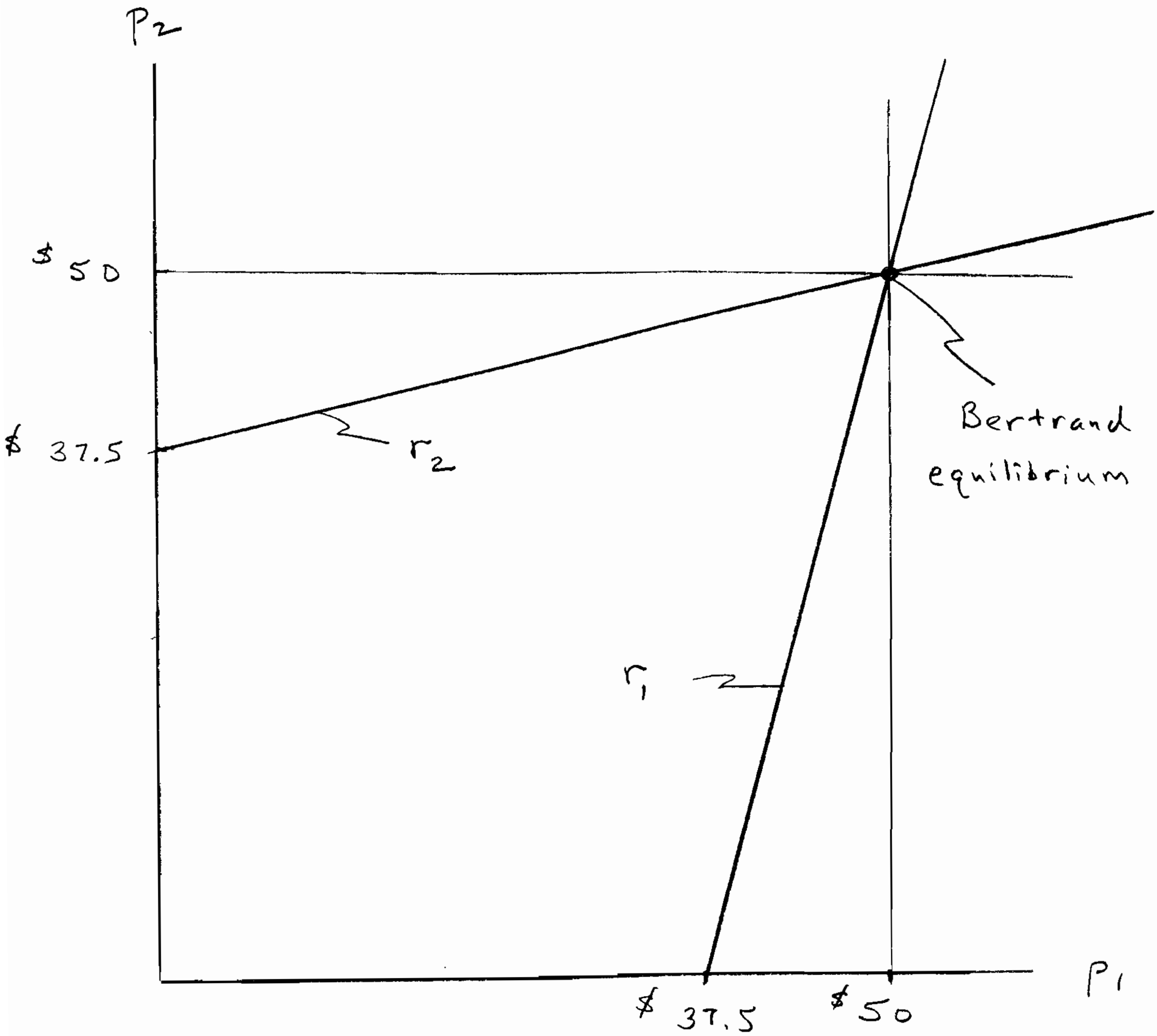



Figure 13.6