Lesson 14
Game Theory

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Version C

1. Introduction

In the last lesson we discussed duopoly markets in which two firms compete to sell a product. In such markets, the firms behave strategically; each firm must think about what the other firm is doing in order to decide what it should do itself. The theory of duopoly was originally developed in the 19th century, but it led to the theory of games in the 20th century. The first major book in game theory, published in 1944, was *Theory of Games and Economic Behavior*, by John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977). We will return to the contributions of Von Neumann and Morgenstern in Lesson 19, on uncertainty and expected utility.

A group of people (or teams, firms, armies, countries) are in a game if their decision problems are interdependent, in the sense that the actions that all of them take influence the outcomes for everyone. Game theory is the study of games; it can also be called interactive decision theory. Many real-life interactions can be viewed as games. Obviously football, soccer, and baseball games are games. But so are the interactions of duopolists, the political campaigns between parties before an election, and the interactions of armed forces and countries. Even some interactions between animal or plant species in nature can be modeled as games. In fact, game theory has been used in many different fields in recent decades, including economics, political science, psychology, sociology, computer science, and biology.

This brief lesson is not meant to replace a formal course in game theory; it is only an introduction. The general emphasis is on how strategic behavior affects the interactions among rational players in a game. We will provide some basic definitions, and we will discuss a number of well-known simple examples. We will start with a description of the prisoners’ dilemma, where we will introduce the idea of a dominant strategy equilibrium. We will briefly discuss repeated games in the prisoners’ dilemma context, and tit for tat strategies. Then we will describe the
battle of the sexes, and introduce the concept of Nash equilibrium. We will discuss the possibilities of there being multiple Nash equilibria, or no (pure strategy) Nash equilibria, and we discuss the idea of mixed strategy equilibria. We will then present an expanded battle of the sexes, and we will see that in game theory, an expansion of choices may make players worse off instead of better off. At the end of the lesson, we will describe sequential move games, and we will briefly discuss threats.


The most well-known example in game theory is the prisoners’ dilemma. (It was developed around 1950 by Merrill M. Flood (1908-1991) and Melvin Dresher (1911-1992) of the RAND Corporation. It was so-named by Albert W. Tucker (1905-1995), a Princeton University mathematics professor.)

Consider the following. A crime is committed. The police arrive at the scene and arrest two suspects. Each of the suspects is taken to the police station for interrogation, and they are placed in separate cells. The cells are cold and nasty. The police interrogate them separately, and without any lawyers present. A police officer tells each one: “You can keep your mouth shut and refuse to testify. Or, you can confess and testify at trial.”

We use some special and potentially confusing terminology to describe this choice. If a suspect refuses to testify, we say that he has chosen to cooperate with his fellow suspect. If a suspect confesses and testifies at trial, we say that he has chosen to defect from his fellow suspect. The reader will need to remember that to “cooperate” means to cooperate with the other suspect, not with the police, and also to remember that to “defect” means to defect from the other suspect.

The officer goes on: “If both of you refuse to testify, we will only be able to convict you on a minor charge, which will result in a sentence of 6 months in prison for each of you. If both of you confess and testify, you will each get 5 years in prison. If one of you refuses to testify (i.e., “cooperates”) while the other confesses and testifies (i.e., “defects”), the one who testifies will go free, and the one who refuses to testify will get a full 10 years in prison.”

The officer concludes: “That’s what we’re offering you, you lowlife hooligan. Think it over. We’ll be back tomorrow to hear what you have to say.”

We now consider this question: given this information, how should a rational suspect behave?
Should the suspects “cooperate” with each other (and tell the police nothing) or should they “defect” from each other (and confess)?

Table 14.1 below shows the prisoners’ dilemma game. In game theory, the people playing the game are called players, so we now refer to our suspects as players. Player 1 chooses the rows in the table, while player 2 chooses the columns. Each of them has two possible actions to choose: “Cooperate” or “Defect.” Each of the four action combinations results in payoffs to each player, in the form of prison time to be served. The outcomes are shown as the vectors in the cells of Table 14.1. The first entry is always the outcome for player 1, and the second is always the outcome for player 2. For instance, if player 1 defects while player 2 cooperates (bottom row, left column of the table), prison time for player 1 is None, and prison time for player 2 is 10 years. Note that these outcomes are “bads” rather than “goods”; each player wants to minimize his outcome.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>6 months, 6 months</td>
</tr>
<tr>
<td>Defect</td>
<td>None, 10 years</td>
</tr>
</tbody>
</table>

Table 14.1: The prisoners’ dilemma.

Each suspect wants to minimize his own jail time. But each must think about what the other suspect will do.

Let us now analyze the problem carefully. Here’s how player 1 thinks about the game. He considers what player 2 might do. If player 2 cooperates, they are in the first column of the table. In this case, player 1 gets 6 months if he cooperates (first row), and no prison time if he defects (second row). Therefore, if player 2 cooperates, player 1 will defect. On the other hand, if player 2 defects, they are in the second column of the table. In this case, player 1 gets 10 years if he cooperates (first row), and 5 years if he defects (second row). Therefore if player 2 defects, player 1 will defect.

We now realize that whatever action player 2 chooses, player 1 will want to defect. We leave it to the reader to do the same type of analysis for player 2, whose payoffs are the second entries
in each of the payoff vectors. When you do this, you will conclude that player 2 will want to defect, whatever action player 1 chooses.

In a game like this, actions that players might take are called strategies. A dominant strategy is a strategy which is optimal for a player, no matter what strategy the other player is choosing. In the prisoners’ dilemma, the best thing for player 1 to do is to defect, no matter what player 2 might do. Therefore “Defect” is a dominant strategy for player 1. Similarly, “Defect” is a dominant strategy for player 2. When a pair of strategies are each dominant for the two players, the pair is called a dominant strategy equilibrium or a solution in dominant strategies. We now know that (Defect, Defect) is a dominant strategy equilibrium in the prisoners’ dilemma. Rational players should choose dominant strategies if they exist; they clearly make sense, since a dominant strategy is the best for a player no matter what the other player is doing.

We conclude that the two suspects should both confess to the police, or defect from each other. Therefore they will each end up with a prison sentence of 5 years. Between the two of them, the total will be 10 years of prison. But this outcome is very peculiar, because if they had both chosen to keep their mouths shut, or cooperate with each other, they would have ended up with prison sentences of only 6 months each, and a total of 1 year between the two of them.

Back in Lesson 11 on perfectly competitive markets, we introduced the reader to Adam Smith’s free market philosophy—his invisible hand theory. In brief, this is the theory that if the market is allowed to operate freely, with each consumer seeking to maximize his own utility and each firm seeking to maximize its own profits, with each of the players in the grand market game ignoring the welfare of all the others and doing the best it can for itself, the outcome will actually be best for society. That is, self-interested consumers and firms in a competitive market will end up maximizing social surplus, the sum of consumers’ and producers’ surplus.

But now note the dramatically different conclusion in the prisoners’ dilemma. In this game, where we are focusing on the outcomes for the two suspects and ignoring the welfare of the police officers, the victims of the original crime, and the rest of society, the obvious and simple measure of social welfare for our two suspects is \(-1\) times the sum of the two prison sentences. (We need the \(-1\) to convert a cost—prison time—into a benefit.) But our analysis above indicates that each player, pursuing his own self-interest, maximizing his own welfare by minimizing his years
in prison, will choose “Defect.” They will end up with a total of 10 years of prison between the two of them. If they had gotten together and determined what would be best for them, and if they had had some way to enforce their agreement, they would have decided on (Cooperate, Cooperate) instead. That would have resulted in a total of 1 year of prison between the two of them. But (Cooperate, Cooperate) is not an equilibrium in the prisoners’ dilemma, and even if they had agreed to keep silent before they were arrested by the police, they would likely have confessed anyway, because of the ever-present incentive to break such an agreement.

The moral of the story is important. In a game, because of the strategic interactions, pursuing individual self-interest may be inconsistent with maximizing social welfare. This matters in evaluating the performance of market institutions in these contexts. We saw in our analysis of duopoly in Lesson 13 that the Cournot equilibrium would not maximize the joint profits of the two duopolists. There are many other examples where strategic interactions result in individual players’ pursuit of private gains producing a loss to the group of players. Famous examples include international arms races, and overutilization of natural resources like fisheries. In these examples, dominant strategies lead to socially undesirable outcomes. The prisoners’ dilemma clearly illustrates the problem—the tension that may exist between self-interest and cooperation. These are two of the key forces in game theory and in reality.

3. Prisoners’ Dilemma Complications: Experimental Evidence and Repeated Games

We have argued above that (Defect, Defect) is a dominant strategy equilibrium in the prisoners’ dilemma. But social scientists have performed experiments to see whether people actually choose the “defect” strategy. (These people are usually university students paid to be experimental subjects in a lab setting.) Often they don’t; often they choose “Cooperate” instead. There are many reasons why this might happen. Subjects may not understand the game, or they simply may not act in the “rational” way that game theorists say they should act. For instance, they might choose “Cooperate” because they believe cooperating is morally preferable to defecting, no matter what the payoffs are. Perhaps game theory is wrong in the sense that it does not correspond to how people actually behave. Another possibility is that the game theory model described above is incomplete. Perhaps we have left something out. This possibility of incompleteness has led some game theorists to expand the model. One of the most important
expansions is the idea of repeated games.

A *one shot game* is a game that is played once. The players choose their strategies, there is an outcome and there are payoffs, and that’s that. A *repeated game* is played over and over. The players choose their strategies, there is an outcome and there are payoffs. Then they do it again. And again. And perhaps again. A repeated game might repeat $n$ times, where $n$ is known beforehand, or it might repeat $n$ times, where $n$ is not known beforehand, or it might repeat an infinite number of times.

Now suppose our prisoners’ dilemma is a repeated game and the players do not know $n$, but think that $n$ might be large. Then a player may choose “Cooperate,” knowing it may cost him in the short run (the current game), but believing that if he chooses “Cooperate,” the other player will be more likely to also choose “Cooperate” in future plays of the game. Similarly, if one player chooses “Defect” in the current game, he may fear that the other player will punish him by defecting in the future. Under certain conditions—if future payoffs matter enough—(Cooperate, Cooperate) is an equilibrium in the repeated prisoners’ dilemma. The moral of the story is that we may see cooperation in situations like the prisoners’ dilemma, where simple game theory indicates we should see defection, not because people are good-hearted or virtuous, but because of a dynamic social contract: “Let’s cooperate with each other now and get good payoffs; for if we don’t, in future periods we’ll punish each other and get bad payoffs.”

Players may also develop retaliatory repeated game strategies affecting their choices within a game, contingent on what has happened in prior periods in the game. One of the simplest is called “tit for tat.” The *tit for tat* repeated game strategy works like this. In the first period of the game, the player chooses “Cooperate.” In any subsequent period, the player looks at his opponent’s action in the previous period of the game. If the opponent chose “Cooperate” in the previous period, then the player chooses “Cooperate” in the current period; if the opponent chose “Defect” in the previous period, then the player chooses “Defect” in the current period. In short, the player matches what his opponent did in the last period of the game. This kind of repeated game strategy might be described as “crazy” or “tough,” but it might also be very effective. Under certain conditions, it can be shown that if player 1 plays “tit for tat,” there may be an equilibrium in which both players are choosing “Cooperate” most of the time. One lesson
here is that it may sometimes be in the interest of people to have reputations as being “crazy” or “tough,” in order to induce beneficial changes in the behavior of others.

The moral of this story is that game theory can sometimes improve its predictions in explaining real-world phenomena by expanding its models.

4. The Battle of the Sexes, and the Idea of Nash Equilibrium

Most games are not as simple to solve as the prisoners’ dilemma. That is, in most strategic situations, players do not have dominant strategies. In general, what each player will want to do will depend on what the other players are doing. Consequently, each player’s conjectures about the behavior of the other players are crucial for determining his own behavior. For example, remember the first duopoly game of the last lesson, and its solution, the Cournot equilibrium \( (y^*_1, y^*_2) \). (Here \( y^*_1 \) is firm 1’s output, and \( y^*_2 \) is firm 2’s.) It is obvious that the Cournot equilibrium is not a dominant strategy equilibrium. If firm 2 decided to flood the market with product and drive the price down to zero, for example, firm 1 would not choose \( y^*_1 \). Rather, firm 1 would produce zero and save its production costs. This shows that producing \( y^*_1 \) is not a dominant strategy for firm 1. The same argument applies to firm 2.

We will now analyze a new game, the battle of the sexes. This was first studied by R. Duncan Luce (1925-) and Howard Raiffa (1924-), in their 1957 book Games and Decisions: Introduction and Critical Survey.

A young woman (player 1) and her boyfriend (player 2) are out on Saturday night, driving in their own cars, on their way to meet each other for an evening together. Since this game was invented long before cellphones were around, they cannot communicate with each other. There are two options that they had talked about previously: a football game and an opera performance. But neither one of them can recall which option they had decided on. They like each other very much, and both would hate to spend the evening without the other. The young woman likes opera much better than football, but her boyfriend likes football better than opera. If the woman ends up at the opera with her boyfriend, her payoff is 3. But her payoff is 0 if she ends up at the opera without him. If the woman ends up at the football game with her boyfriend, her payoff is 1. But her payoff is 0 if she ends up at the football game without him. Similarly for the young man, if he ends up at the football game with her, his payoff is 3; if he ends up at the opera with
her, his payoff is 1; and if he ends up at either place without her, his payoff is 0.

Table 14.2 shows the game. The rows of the table are the woman’s possible strategies, and the columns are the man’s. In other words, the woman chooses the row, and the man chooses the column. Each vector in each cell of the table shows the payoffs to the two players. For instance, if both of them choose football, they are in the first row, first column cell of the table. The payoff to the woman is then 1, and the payoff to the man is 3. Note that these payoffs, unlike the payoffs in the prisoners’ dilemma game, are “goods” rather than “bads”; each player want to maximize rather than minimize her/his outcome.

<table>
<thead>
<tr>
<th>Woman</th>
<th>Man</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>Football</td>
</tr>
<tr>
<td>1, 3</td>
<td>0, 0</td>
</tr>
<tr>
<td>Opera</td>
<td>Opera</td>
</tr>
<tr>
<td>0, 0</td>
<td>3, 1</td>
</tr>
</tbody>
</table>

Table 14:2 The battle of the sexes.

What predictions can we make about this game? First of all, note that there are no dominant strategies. For either player, “Football” is better if she/he expects the other to choose “Football,” but “Opera” is better if she/he expects the other to choose “Opera.”

The standard equilibrium concept in the battle of the sexes is the Nash equilibrium, named for the famous 20th century economist, mathematician, and game theorist John Nash (1928-). A Nash equilibrium is a pair of strategies, one for each player, such that player 1’s strategy is the best for her given player 2’s strategy, and such that player 2’s strategy is the best for him given player 1’s strategy. Each player’s strategy is a best response to the other’s.

The reader should note that a Cournot equilibrium in a duopoly model is a Nash equilibrium, and a Bertrand equilibrium in a duopoly model is also a Nash equilibrium in the corresponding duopoly game. Moreover, any dominant strategy equilibrium is a Nash equilibrium. For example, (Defect, Defect) in the prisoners’ dilemma is also a Nash equilibrium. This is because a dominant strategy for a player is always a best response for that player; therefore it is the best response when his opponent is playing his dominant strategy. But the reverse doesn’t hold; and there will
generally be Nash equilibria in a game that are not dominant strategy equilibria. Remember that there are no dominant strategies in our battle of the sexes, and therefore no dominant strategy equilibria. What about the existence of Nash equilibria in the battle of the sexes?

There are two Nash equilibria in the battle of the sexes: (Football, Football) with payoffs (1,3), and (Opera, Opera) with payoffs (3,1). Here is why (Football, Football) is a Nash equilibrium. (The argument for (Opera, Opera) is entirely symmetric.) If player 1 expects player 2 to drive to the football game, that’s what she will choose as well, because a payoff of 1 is greater than a payoff of 0. And if player 2 expects player 1 to drive to the football game, that’s what he will choose as well, because a payoff of 3 is greater than a payoff of 0.

Each Nash equilibrium is a theory of how the game should be played, consistent with assumed rationality of the players and the mutual knowledge of that rationality. It seems plausible to predict that player 1 and her boyfriend will end up at a Nash equilibrium in this game, or at least that they ought to end up at a Nash equilibrium. It is certainly the case that at the planning stages of the game, when the players are talking to each other about going to a football game or going to the opera, they are only considering going to the same event together. That is, these rational players, in planning this game, would agree that the non-Nash outcomes are undesirable, and that the Nash equilibria, even though one is inferior to the other in each player’s eyes, are reasonable in the sense that neither player would want to break an agreement to be at such an outcome.

5. Battle of the Sexes Complications: Multiple or No Nash Equilibria, and Mixed Strategies

From the battle of the sexes, we see that there may be multiple Nash equilibria. So the Nash equilibrium concept may have some predictive power—(Football, Football) and (Opera, Opera) seem more likely than (Football, Opera) and (Opera, Football)—but it may not point to a unique outcome.

Moreover, in this game, the players may end up at a non-Nash outcome by accident, if not by intent. That is, even if our young woman and her boyfriend know exactly what their preferences are, and are completely informed about Table 14.2 and the Nash equilibria in that table, they just don’t remember which event they had planned to attend, and they have no cellphones with which to communicate. Therefore they may end up apart, even though their feelings toward each
other, and the power of Nash reasoning, say they should be together.

And things may get even trickier. There may be no equilibria of the kind we have been describing. Consider the following *strangely modified battle of the sexes*: Let the two players have the same payoffs as before when they are *coordinated*. That is, when they choose (Football, Football) and (Opera, Opera), the payoffs are (1, 3) and (3, 1), respectively. But when they are *miscoordinated*, and choose (Football, Opera) or (Opera, Football), they won’t get payoffs of (0, 0). Rather, they will get the following: at (Football, Opera) the payoffs are (4, −4), and at (Opera, Football) the payoffs are (2, −2).

Here’s a possible explanation for these payoffs. At the miscoordinated pairs of strategies, the totals of the payoffs to the young woman and her boyfriend are zero, as they were previously. The boyfriend’s payoffs are simple to explain. He’s happiest (payoff 3) when they are together at the football game, less happy (payoff 1) when they are together at the opera, even less happy (payoff −2) when he’s alone at the football game, and miserable (payoff −4) when he’s alone at the opera. It is more difficult to explain the young woman’s preferences, perhaps because women are more complex than men. When she and her boyfriend are together, she is happier at the opera (payoff 3) than at the football game (payoff 1). However if they are miscoordinated and she is at the football game by herself, she is happiest (payoff 4). This surprising payoff is because she feels that although she loves opera and her boyfriend, it would be really good for her to be forced to learn something about football, and for him to be forced to learn something about opera. If she is at the opera by herself, her payoff is 2, not as good as being at the opera with him (payoff 3), but better than being at the football game with him (payoff 1). Payoffs in the strangely modified battle of the sexes are shown in Table 14.3 below.

<table>
<thead>
<tr>
<th>Woman</th>
<th>Man</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Football</td>
</tr>
<tr>
<td>Football</td>
<td>1, 3</td>
</tr>
<tr>
<td>Opera</td>
<td>2, -2</td>
</tr>
</tbody>
</table>

Table 14.3: The strangely modified battle of the sexes.

When we examine the table of payoffs in Table 14.3, we see the following. From the upper
left cell, player 1 would want to move down to the lower left cell. From the lower left cell, player 2 would want to move right to the lower right cell. From the lower right cell, player 1 would want to move up to the upper right cell. From the upper right cell, player 2 would want to move left to the upper left cell. In short, at every pair of strategies, one of the players would be unhappy and would want to change her or his strategy. Therefore, at least based on our definition of Nash equilibrium to this point, there is no Nash equilibrium in this game.

In fact, our definition of Nash equilibrium up to now has assumed that a player can only choose a single strategy with certainty. Player 1, for instance, can choose either “Football” or “Opera.” If she chooses “Football,” she goes to the football game for sure. Going to the football game for sure is called a pure strategy. The games we have been discussing to this point allow only pure strategies. Player 1 can go to the opera, or she can go to the football game. That’s it.

But there is another way to play games like this. Players might make random choices over pure strategies. For instance, player 1 might decide: “I’m going to flip a coin, and go to the football game if it’s heads, and to the opera if it’s tails.” This means she decides: “I’ll choose “Football” with probability 1/2, and I’ll choose “Opera” with probability 1/2.” This is an example of what is called a mixed strategy. More formally, if there are two pure strategies, say $S_1$ and $S_2$, a mixed strategy is a pair of probabilities, say $p_1$ and $p_2$, chosen by the player and summing to 1, with the player choosing $S_1$ with probability $p_1$ and choosing $S_2$ with probability $p_2$. (Note that any pure strategy is also a mixed strategy, but not vice versa. For example, the pure strategy $S_1$ is the same as the mixed strategy over $S_1$ and $S_2$ with $p_1 = 1$ and $p_2 = 0$.) A pure strategy Nash equilibrium is a Nash equilibrium in which players use pure strategies. A mixed strategy Nash equilibrium is a Nash equilibrium in which players use mixed strategies.

What we have shown with the strangely modified battle of the sexes is that there may be no pure strategy Nash equilibrium in a game. In a famous paper written in 1951, John Nash (1928-) proved that under general conditions, any game with a finite number of pure strategies must have at least one mixed strategy equilibrium. It follows that our strangely modified battle game must have a mixed strategy Nash equilibrium, even though it doesn’t have a pure strategy equilibrium. In this lesson we will not discuss how one might find the mixed strategy equilibrium which we know, thanks to Nash, must exist.
In the rest of this lesson we will return to our focus on pure strategies and pure strategy equilibria.

6. The Expanded Battle of the Sexes, When More Choices Make Players Worse Off

In the decision problem for an individual consumer or firm, the expansion of the set of feasible actions has a clear effect—the decision maker cannot end up worse off than before, and will likely end up better off. Consider, for example, the basic consumer choice model. When the budget set expands, whether because of an increase in income with prices fixed, or because of a fall in prices with income fixed, the consumer will generally be better off, and will definitely not be worse off. In this section, we shall see that this basic property—expansion of the choice set is a good thing for the decision maker—may not hold in a strategic situation.

We now turn to an expanded battle of the sexes. Here is the story. After the original battle of the sexes described above (not the strangely modified version), some weeks pass. Our couple gets into a fight. They are mad at each other, but they are still together. Another Saturday rolls around, and it’s time for another date. The old options of football and opera are still there, and our young woman and her boyfriend have exactly the same feelings they used to have about those options. But there is a new option available to them: the player can stay at home, and deliberately stand up her/his date. (We are assuming the two live separately, so if one stays at home, the other doesn’t immediately observe it.) If the woman stays at home and the boyfriend goes out, we will assume she gets a payoff of 2. (This is the satisfaction of hurting her boyfriend.) And we will assume the boyfriend gets a payoff of -1. (This is the pain from discovering he was deliberately stood up.) Similarly, if the boyfriend stays at home and she goes out, we assume he gets a payoff of 2 and she gets a payoff of -1. If they both stay at home, we assume a payoff of 0 to each.

Table 14.4 below shows the table of payoff vectors. Note that the payoffs are exactly the same as they used to be for the four pairs of strategies in the original battle of the sexes, as shown in Table 14.2. What’s new are the third row in Table 14.4, based on player 1 staying at home, and the third column, based on player 2 staying at home. Everything that player 1 and player 2 used to be able to do, they can still do. But now they have more options. The table showing the possible payoff vectors is now 3 by 3 instead of 2 by 2; it has 9 cells instead of 4. Each of the
new cells looks worse for both players than at least one of the old cells.

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{Man} & \text{Woman} \\
\hline
\text{Football} & 1, 3 & 0, 0 & -1, 2 \\
\text{Opera} & 0, 0 & 3, 1 & -1, 2 \\
\text{Stay Home} & 2, -1 & 2, -1 & 0, 0 \\
\hline
\end{array}
\]

Table 14.4: The expanded battle of the sexes.

We have expanded the options available to the two players. But whatever was available to them in the past is still available. What are the effects of this expansion of choices?

First, it is easy to see that the old Nash equilibria, of the original battle of the sexes, are no longer Nash equilibria in this new game. Take, for instance, the pair of strategies (Football, Football). Now if the woman expects her boyfriend to drive to the football game, her best response is no longer to drive to the football game and meet him there, which would have given her a payoff of 1. Rather, she will stay at home, which will give her a payoff of 2. Similarly, the pair of strategies (Opera, Opera) is no longer a Nash equilibrium, since now the man (whose payoff is 1 at (Opera, Opera)) prefers to stay home, which will give him a payoff of 2.

In fact, the only Nash equilibrium in the expanded battle of the sexes is (Stay Home, Stay Home), which has payoffs of 0 for both players. Let’s check this. If the woman expects her boyfriend to stay home, she looks at the third column of Table 14.4. She gets a payoff of -1 if she goes to the football game, a payoff of -1 if she goes to the opera, and a payoff of 0 if she stays home also. Her best response is therefore to stay home. The argument is symmetric for the man. If he thinks she is staying home, his best response is to stay home also. Therefore (Stay Home, Stay Home) is a Nash equilibrium. It is easy to see that any of the pairs of strategies where one person goes out and the other person stays at home cannot be a Nash equilibrium. We’ll leave it to the reader to check this.

The addition of a new strategy has had a major effect in the battle of the sexes. It has demoted the original pair of Nash equilibria—they are no longer Nash equilibria. It has created a new
Nash equilibrium, which is now the only equilibrium in the game. Moreover, at the new Nash equilibrium, the payoff vector \((0, 0)\) is worse for both players than the original Nash equilibrium payoff vectors of \((1, 3)\) and \((3, 1)\). The expansion of choices has had the effect of making both players worse off at the Nash equilibria.

What produced this strange result? The addition of the new choice led to a very different strategic situation, which undermined the original Nash equilibria and paradoxically elevated a new, and worse, equilibrium.

7. Sequential Move Games

All the games we have presented so far are simultaneous move games. This means that (at least in theory) the two players choose their strategies at the same time, each one not knowing what the other is choosing. Then there is an outcome, and payoffs are made. (The repeated games we mentioned in Section 3 above were sequences of simultaneous move games, with payoffs made at the end of each game in the sequence.)

However there are other games in which time plays a crucial role, where one player moves first and is observed by the other player who moves second, after which payoffs are made. And there are games where the players make a sequence of moves, alternating turns, with each player observing the other player’s move at each step of the process, and with payoffs made at the end. These games are called sequential move games or sequential games. We will discuss such games in this section.

In sequential move games, the conventional wisdom is that there is a first-mover advantage. It is better to move first, because a first move sets the tone for the rest of the game, and the first mover can create the kind of play that she or he wishes. In the game of tic-tac-toe for instance, the first mover seems to have an advantage because he has 9 squares available at his first move, whereas the second mover only has 8 squares. (Studies indicate there is a first-mover advantage in tic-tac-toe for players of average skill—who make errors—but not for expert players. A tic-tac-toe game between experts should result in a tie.) In chess, there is serious debate about whether or not white has a first-move advantage over black. There are studies that indicate white wins a slightly higher proportion of tournament games than black. Some chess experts claim that perfectly played games should result in a draw; others claim that perfectly played games should
result in a win for white.

In the following examples, we will show that in theory, sequential games do not necessarily provide an advantage to the first mover. Whether there exists a first-mover or a second-mover advantage will depend on the specifics of the game.

We now consider a sequential version of what is called the matching pennies game. Generally, in a matching pennies game, two players each place a penny on the table. If the pennies “match,” meaning they are both heads or both tails, a dollar is paid by one of the players to the other player. If the pennies “do not match” (one is a head and the other a tail), the dollar transfer goes in the opposite direction. This can be a simultaneous move game (in which case it is like the ancient and familiar odds-and-evens game) or a sequential move game. We will now consider the sequential move game.

Assume that player 1 moves first, and must put his penny on the table, either face up (“Heads”) or face down (“Tails”). Player 2 observes this. Then she moves, and puts down her penny, either face up (“Heads”) or face down (“Tails”). The rules of the game require that player 1 pay $1 to player 2 if the pennies match, and that player 2 pay $1 to player 1 if they do not match. Figure 14.1 shows the game in the form of a game tree.

A game tree is a diagram with connected nodes and branches. Time flows from left to right in the diagram. At the farthest left is a node, at which the first player to move (player 1 in this case) chooses a strategy. Each strategy is represented by a branch to the right. At the end of each of those branches are new nodes, at which the second player to move chooses her actions. The ultimate payoff vectors appear at the very end of the sequence of nodes and branches. In Figure 14.1, for example, the uppermost sequence of nodes and branches can be read as follows. Player 1 starts the game and chooses heads. Then player 2 goes and chooses heads. Then the game ends, with payoffs to players 1 and 2 of -1 and +1, respectively.

Figure 14.1: Draw a game tree, with the initial decision node labeled “Player 1” with two branches labeled “Heads” and “Tails.” Each of these two branches arrives at a decision node labeled “Player 2.” At each of 2’s decision nodes are two branches labeled also “Heads” and “Tails.” Payoff vectors at the end of the four branches of the tree are: (-1,1), (1,-1), (1,-1) and (-1,1).
Caption of Fig. 14.1: The sequential version of matching pennies.

To solve a sequential game like this, we apply a procedure called *backward induction*. This procedure assumes that at each decision node, each player will behave optimally, given his or her theory about how the players will behave at nodes farther in the future. To solve the game with backward induction, we go to the last decision nodes in the game tree, the ones farthest in the future (and farthest to the right in the game tree). We determine the optimal action (or actions) for that player making the decision at that point in time. Having done so, we go backwards in time (and to the left in the game tree) and determine the optimal action (or actions) at the previous set of decision nodes. We repeat this until we have gone all the way back in time (and all the way to the left in the game tree), and determined the optimal action at the first node of the game, for the first mover.

Let’s do this in Figure 14.1. We go to the last decision nodes, the ones for player 2. At the upper node (which follows player 1’s choice of “Heads”), if player 2 chooses “Heads,” her payoff is +1. If she chooses “Tails,” her payoff is -1. Therefore she chooses “Heads.” At the lower decision node (which follows player 1’s choice of “Tails”), if player 2 chooses “Tails,” her payoff is +1. If she chooses “Heads,” her payoff is -1. Therefore she chooses “Tails.” We see at this stage that player 2 is always going to win the dollar. We now move to the left and decide what player 1 should do at the first decision node. The answer is that it doesn’t matter, he can choose “Heads” or “Tails.” The outcome is the same to him in either case. Either one of these leads to the payoff vector (-1,+1). In short, in this game, player 2, the second mover, will win the dollar. This game has a clear second-mover advantage. This shows that whether there is a first-mover or a second-mover advantage in a game depends on the specifics of the game.

We will complete the discussion of the sequential matching pennies game with an observation about the distinction between strategies and actions. In game theory, a strategy is a complete contingent plan of the actions which a player will play in a game. If it is a simultaneous move game, where the actions all take place at one point in time, a strategy coincides with an action. In a sequential move game, a strategy does not necessarily coincide with an action because a player who moves later in the game can make his actions contingent on the history of actions before his. To be clear, in the sequential matching pennies game, player 1 has only two strategies, which
coincide with his actions: “Heads” and “Tails.” But player 2 has four strategies: “Always Heads: After Heads and After Tails, Play Heads,” “Always Tails: After Heads and After Tails, Play Tails,” “Matching: After Heads, Play Heads, and After Tails, Play Tails,” and “Not Matching: After Heads, Play Tails, and After Tails, Play Heads.” Therefore, there are two backward induction strategy solutions to this game: player 1 chooses “Heads” and then player 2 chooses “Matching,” and player 1 chooses “Tails” and then player 2 chooses “Matching.” This more careful analysis still leads to the conclusion that the second player in the game will match the action of the first player, and will win the dollar.

We now consider a slightly different game, which we will call a duopoly sequential competition game. The two players are now two firms in a duopoly market. Firm 1 moves first and can produce a “High” output or a “Low” output. After firm 2 observes firm 1’s choice of output, it responds by also choosing either “High” or “Low.” Assume that the payoffs to the firms, that is, profits, are \((\pi_1, \pi_2) = (-1, -1)\) if both firms choose “High” because the market is inundated with the product and the price falls below average cost. If both firms produce “Low,” profits are \((\pi_1, \pi_2) = (2, 2)\). Finally, if one firm produces “High” and the other produces “Low,” assume that the firm with the higher output ends up with profit of 3, while the firm with the low output has profit of 1. Figure 14.2 represents this game in a game tree.

Figure 14.2: Draw a game tree, with the initial decision node labeled “Firm 1” with two branches labeled “High” and “Low.” Each of these two branches arrives at a decision node labeled “Firm 2.” At each of 2’s decision nodes, two branches, labeled also “High” and “Low.” Payoff vectors at the end of the four branches of the tree are: \((-1, -1)\), \((3, 1)\), \((1, 3)\) and \((2, 2)\).

Caption of Fig. 14.2: A duopoly sequential competition game.

Applying the backward induction procedure to this game, we go first to firm 2’s decision nodes. If firm 1 has produced “High,” firm 2 will produce “Low” because 1 is greater than -1. And if firm 1 has produced “Low,” firm 2 will respond with “High” because 3 is greater than 2. Now we go back to firm 1’s decision node. Firm 1 knows that firm 2 will do the opposite of what it has done. If firm 1 chooses “High,” it will end up with a payoff of 3. If firm 1 chooses “Low,” it will end up with a payoff of 1. Therefore firm 1 will choose “High.” Firm 2 will respond with “Low,” and the ultimate profits will be \((\pi_1, \pi_2) = (3, 1)\).
As you can easily see, there is a first-mover advantage in this game. The game is completely symmetric in payoffs, and so, if the roles of firm 1 and 2 were reversed (with firm 2 moving first and firm 1 moving second), we would end up with a similar outcome, with the first mover choosing “High”, and the second mover responding with “Low.” With the roles reversed, the payoff vector would be \((\pi_1, \pi_2) = (1, 3)\). This game should remind the reader of the Stackelberg solution to the duopoly model.

8. Threats

We conclude this lesson by briefly discussing threats. A threat is an announcement made by a player at the beginning of a sequential move game, indicating that at some node, at some point in time, he will depart from what is rational in order to punish the other player. The sequential move game framework can help us to evaluate the credibility of threats. For instance, in the duopoly sequential competition game of the section above, firm 2 could try to change the outcome of (High, Low) by threatening firm 1 as follows: “No matter what you do, my plan is to produce “High.” Therefore if you decide to produce “High,” we will actually end up with a payoff vector of (-1,-1). I won’t do what you think I ought to do. I will take us both down if you produce “High.””

Obviously, if firm 1 believes the threat, it should produce “Low,” for which the payoff vector is (1,3). A payoff of +1 is much better than a payoff of -1. But in a sequential move game like this, especially if it is played just one time, firm 1 probably should not believe firm 2’s threat. The reason is this: if firm 2 made the threat before the game started, and if firm 1 ignored the threat at the first move, firm 2 would make itself better off when it moves by not carrying through on its threat. If it drops the threat, it ends up with +1. If it carries through on its threat, it ends up with -1. So threats like this seem less credible from the vantage point of the backward induction procedure.

Of course life may be more complicated if games are played over and over, or if people (or firms) play games with different partners, and develop reputations that spread out to other players. If a game is played over and over between two players, an aggressive player may carry out threats in the initial games, so that his playing partner comes to believe that he will carry out his threats, no matter how self-destructive they may be. In this case, his partner becomes
trained to give in to his threats. Or, if he plays with many different players who talk to each other, an aggressive player may want one player to see that he carries out his threats, so that word gets around.

Here is a final observation about some very large threats. For most of the second half of the 20th century, there was a cold war with the United States on one side and the Soviet Union on the other. In this cold war, the two superpowers accumulated large stockpiles of nuclear weapons. Those stockpiles of weapons still exist. The superpowers threatened each other with those weapons. One reason the cold war never became a hot war was the two-way threat of mutual assured destruction, abbreviated MAD, also called nuclear deterrence or massive retaliation. The idea of the mutual assured destruction game was this. If one of the superpowers attacked the other, even in an indirect, non-devastating way, the superpower that had been attacked would retaliate with a massive nuclear strike. For instance, if the Soviet Union invaded (Western) Europe, the United States would launch nuclear weapons against the Soviet Union. This retaliation would lead to a world-wide nuclear war, effectively destroying both superpowers.

The MAD game would have been played just one time. Our comments above suggest that the Soviet Union’s threats against the United States, and the United States’ threats against the Soviet Union, may have all been hollow threats. Or maybe they weren’t. Or maybe the threats were so huge that even if they were unbelievable, neither side could dare to test them.

9. A Solved Problem

The Problem

Consider the following coordination game. There are two players and two strategies available to each player: A and B. The payoffs in the first row (corresponding to player 1 choosing A) are \((a, a)\) and \((0, 0)\). The payoffs in the second row (corresponding to player 1 choosing B) are \((0, 0)\) and \((1, 1)\).

(a) Draw the 2 × 2 payoff matrix.

(b) For what values of \(a\) is \((A, A)\) a dominant strategy equilibrium?

(c) For what values of \(a\) is \((B, B)\) a dominant strategy equilibrium?
(d) Can you find the Nash equilibria of the game as a function of the parameter $a$?

**The Solution**

(a) The payoff matrix is shown in Table 14.5 below.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$a$, $a$</td>
<td>0, 0</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Table 14.5: When is (A, A) a dominant strategy equilibrium? When is (B, B) a dominant strategy equilibrium?

(b) (A, A) can never be a dominant strategy equilibrium, no matter what $a$ is. For (A, A) to be a dominant strategy equilibrium, A would have to be a dominant strategy for both players. But if player 2 is playing B (right column), player 1 is better off with B (payoff 1) than with A (payoff 0). So no matter what $a$ is, playing A cannot be a dominant strategy for player 1. (Similar comments apply to player 2.) Therefore (A, A) cannot be a dominant strategy equilibrium.

(c) If $a \leq 0$, then B is a dominant strategy for player 1. If player 2 chooses A (left column), player 1 is at least as well off at B (payoff 0) as he is at $a$ (payoff $a$); and if player 2 chooses B (right column), player 1 is better off at B (payoff 1) than at A (payoff 0). Similarly, B is a dominant strategy for player 2. If player 1 chooses A (top row), player 2 is at least as well off at B (payoff 0) as at A (payoff $a$); and if player 1 chooses B (bottom row), player 2 is better off at B (payoff 1) than at A (payoff 0). Since B is a dominant strategy for player 1, and B is a dominant strategy for player 2, (B, B) is a dominant strategy equilibrium. Since it is a dominant strategy equilibrium, it is also a Nash equilibrium.

(d) If $a \geq 0$, then (A, A) is a Nash equilibrium. At (A, A), both players compare the payoff $a$ to the payoff 0, and since $a \geq 0$, (A, A) is a Nash equilibrium. But (B, B) is also a
Nash equilibrium when \( a \geq 0 \). This is because no matter how big \( a \) might be, at (B, B), the payoff for both players is 1, and a deviation by either player 1 or player 2 (not both simultaneously) would reduce that player’s payoff to 0. However, if \( a < 0 \), then (B, B) is the only Nash equilibrium.
Exercises

1. Consider the game of chicken with two players. If both players play “Macho,” each of them gets a payoff of 0. If both players play “Chicken,” each of them gets a payoff of 6. If one player plays “Macho” and the other plays “Chicken,” the one who plays “Macho” gets a payoff of 7 and the one who plays “Chicken” gets a payoff of 2.

(a) Draw the payoff matrix.
(b) Does either player have a dominant strategy in this game?
(c) Find the Nash equilibrium or equilibria.

2. Jack and Jill want a treehouse to play in. They have to decide simultaneously whether to build or not to build. Each individual who builds bears a cost of 3. They both have access to the treehouse once it is built. If only one of them builds the treehouse, they each derive a utility of 2. If both of them build the treehouse, they each derive a utility of 4 (presumably the treehouse is more elaborate because two heads are better than one). If the treehouse is not built, they each derive a utility of 0.

(a) Draw the payoff matrix.
(b) What is Jack’s strategy? What is Jill’s strategy? What is the Nash equilibrium or equilibria?
(c) Does this game resemble the prisoner’s dilemma, the battle of the sexes, or chicken? Explain.

3. Sam and Dan are twins who like playing tricks on each other. Sam is deciding whether to take Dan’s blanket. Sam has a utility of 0 if he doesn’t take Dan’s blanket. If Sam takes Dan’s blanket, there is a possibility of Dan retaliating by taking Sam’s pillow, thereby earning Sam a utility of −5. If Dan doesn’t retaliate, Sam gets a utility of 5. Dan has a utility of 10 if Sam doesn’t take his blanket. If Sam takes his blanket, Dan’s utility is -10. Dan’s utility changes by X if he retaliates.
(a) Draw the game tree.

(b) For what values of \( X \) would we observe Sam taking Dan’s blanket in the backward induction equilibrium?

4. Consider the following sequential strategic situation, called the centipede game. The game has 100 stages. There are two players who take turns making decisions, starting with player 1. At stage \( t = 1, \ldots, 99 \), player 1 (if the stage is odd) or player 2 (if it is even) chooses whether to “Terminate the game” or to “Continue the game.” If the game is terminated at stage \( t = 1, \ldots, 99 \), the player terminating the game receives a payoff of \( t \), while the other player receives a payoff of zero. Finally, at stage \( t = 100 \), player 2 chooses between action A with a payoff of 99 for each player, or action B with a payoff of zero for player 1 and a payoff of 100 for player 2.

(a) Draw the game tree for this strategic situation (the name of the game will become apparent then).

(b) What is the backward induction solution to this game?

5. Two players take turns choosing a number between 1 and 10, inclusive. The number is added to a running total. The player who takes the total to 100 (or greater) wins.

(a) What is the backward induction solution to this game? Map out the complete strategy.

(b) Is there a first-mover or a second-mover advantage in this game?

6. Consider a Bertrand duopoly with a homogeneous good, as in the first part of Section 6 of Lesson 13. Assume the market demand curve is \( y = y_1 + y_2 = 1 - p \), where \( p \) is the relevant market price, \( y \) is the total amount demanded at that price, and \( y_1 \) and \( y_2 \) are the output levels for firms 1 and 2. Assume the firms’ cost functions are \( C(y_i) = \frac{1}{2}y_i \) for \( i = 1, 2 \). The rules of the pricing game are as follows. The firms must each simultaneously name a price in the interval \([0, 1]\). If the prices are different, the firm with the lower price sells all the units demanded at that price, while the other firm sells nothing. If they name the same
price, the amount demanded at that price is split equally between the two firms. Show that there is a unique Nash equilibrium, and find the equilibrium price and quantities.

Hint: Note that calculus cannot be used to solve this problem, because the firms’ profit functions are not continuous in the price variables. For instance, \( \pi_1(p_1, p_2) \) is not continuous at \( p_1 = p_2 = p \).
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