Customers are uniformly distributed between zero and one. Each customer chooses the closest seller. Three are three sellers X, Y and Z. Player X chooses its place between zero and one first. Player Y chooses next. Player Z chooses the place given the places of X and Y. The places of X, Y and Z are denoted by x, y and z.

- **Best Response of Z Given** (x, y)

  Start with the case: $x < y$.

  Given $x < y$, Z has three strategies denoted by 1', 2' and 3'.

  1. Strategy 1': $z = x$.
  2. Strategy 2': $x < z < y$.
  3. Strategy 3': $z = y$.

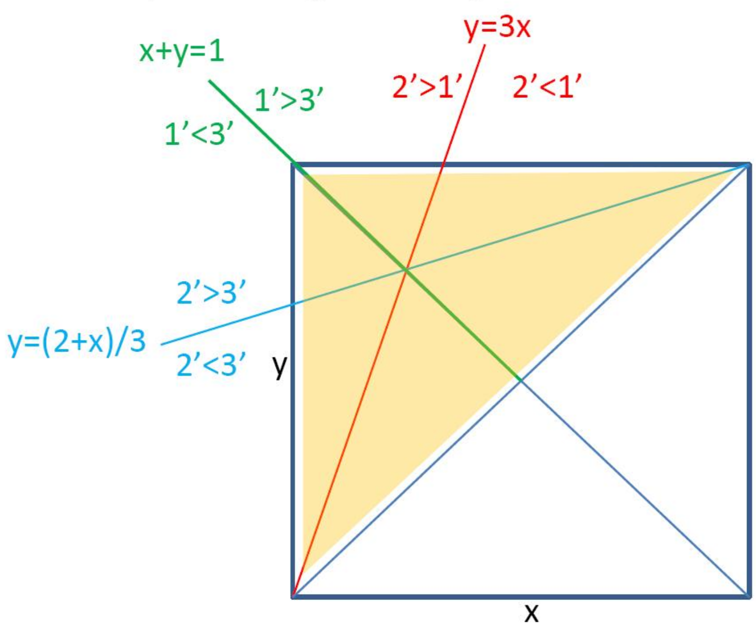
  The corresponding payoffs will be

  1. $P_z^1(z = x) = x$
  2. $P_z^2(z \in (x, y)) = (y - x)/2$
  3. $P_z^3(z = y) = 1 - y$.

  Lemma 1. *$P_z^2 > P_z^1$ when $y > 3x$. $P_z^3 > P_z^1$ when $y < -x + 1$. $P_z^3 > P_z^2$ when $y < (x + 2)/3$.*

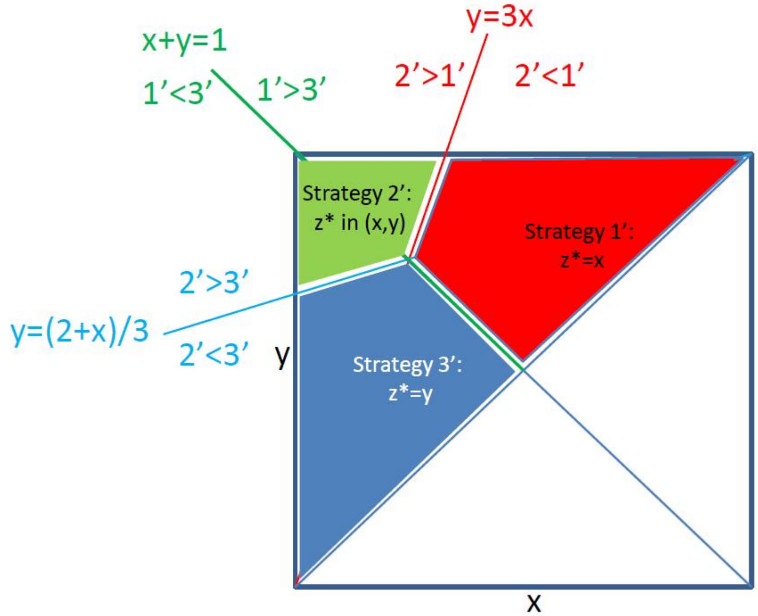
  Lemma 1 is described in Figure 1.

Figure 1: Compare Strategies Given y>x



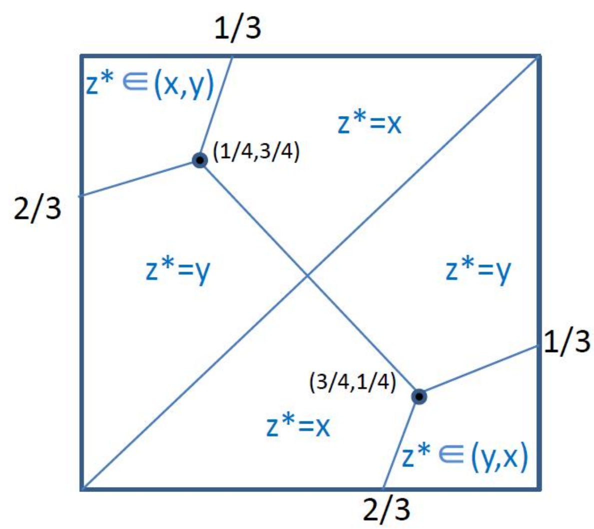
From Lemma 1, we can find the best response of Z, $z^*(x, y)$, which is described in Figure 2.

Figure 2: Indentify Best Response of Z Given y>x



In the same way, we can prove the best response of Z given $x > y$. The full diagram for $z^*(x, y)$ is displayed in Figure 3.

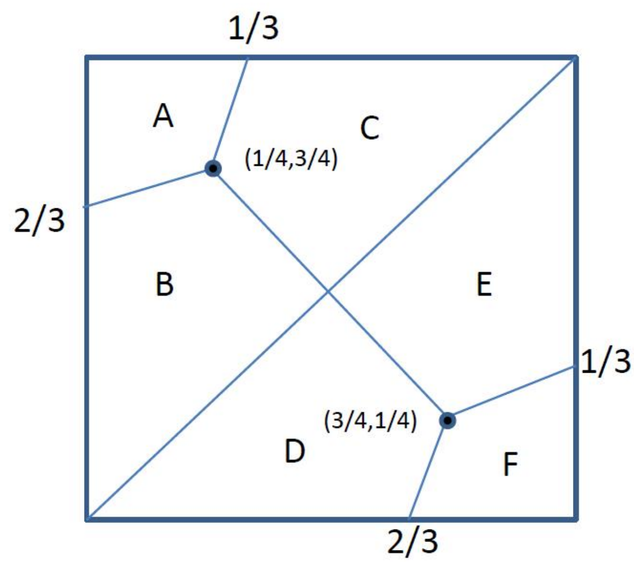
Figure 3: Indentify Best Response of Z Given any (x,y)



- **Best Response of Y Given x and Knowing** $z^*(x, y)$

  Let us denote regions identified with different $z^*(x, y)$ by A, B, C, D, E and F, as displayed in Figure 4.

Figure 4: Denote Each Region by A, B, C, D, E and F



In regions A and F, z^* is located between x and y. In regions B and E, $z^* = y$. In regions C and D, $z^* = x$. Then, the expected payoff for player Y, $E(P_y)$, in each region is:

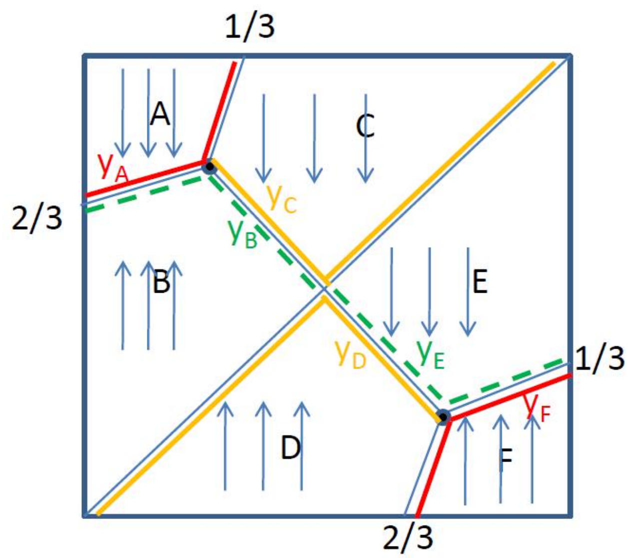
1. $E(P_y^A) = 1 - y + (y - x)/4 = -(3y + x)/4 + 1$
2. $E(P_y^B) = (y - x)/2$
3. $E(P_y^C) = 1 - y + (y - x)/2 = -(x + y)/2 + 1$
4. $E(P_y^D) = y + (x - y)/2 = (x + y)/2$
5. $E(P_y^E) = (x - y)/2$
6. $E(P_y^F) = y + (x - y)/4 = (x + 3y)/4.$

Out of the payoffs, we can prove the following lemma.

Lemma 2. *In regions A, C and E, the payoff for player Y is greater with the smaller y given x. In regions B, D and F, the payoff for player Y is greater with the bigger y given x.*

Using Lemma 2, with x given, the level of y in each region that provides the greatest payoff for Y is denoted in Figure 5. In regions A, C and E, the bottom boundaries (y_A, y_C and y_E) provide the greatest payoff for Y given x. In regions B, D and F, the upper boundaries (y_B, y_D and y_F) provide the greatest payoff for Y given x.

Figure 5: Best Response of Y in Each Region



Now, for any given x, let us find the best response of Y, $y^*(x|z^*(x,y))$.

(1) **Given** $0 \le x \le 1/4$ Note that $y_A = (x+2)/3 + \epsilon$, $y_B = (x+2)/3 - \epsilon$ and $y_D = x - \epsilon$. We have

$$E(P_y^A)(y = y_A) = (1-x)/2 - 3\epsilon/4 \quad (1)$$
$$E(P_y^B)(y = y_B) = (1-x)/3 - \epsilon/2 \quad (2)$$
$$E(P_y^D)(y = y_D) = x - \epsilon/2. \quad (3)$$

Thus, $y^* = y_A$.

(2) **Given** $1/4 < x \le 1/3$ Note that $y_A = 3x + \epsilon$, $y_C = 1 - x + \epsilon$, $y_B = 1 - x - \epsilon$, and $y_D = x - \epsilon$. We have

$$E(P_y^A)(y = y_A) = 1 - 5x/2 - 3\epsilon/4 \quad (4)$$
$$E(P_y^C)(y = y_C) = 1/2 - \epsilon/2 \quad (5)$$
$$E(P_y^B)(y = y_B) = 1/2 - x - \epsilon/2 \quad (6)$$
$$E(P_y^D)(y = y_D) = x - \epsilon/2 \quad (7)$$

Thus, $y^* = y_C$.

(3) **Given** $1/3 < x < 1/2$ Note that $y_C = 1 - x + \epsilon$, $y_B = 1 - x - \epsilon$, and $y_D = x - \epsilon$. We have

$$E(P_y^C)(y = y_C) = 1/2 - \epsilon/2 \quad (8)$$
$$E(P_y^B)(y = y_B) = 1/2 - x - \epsilon/2 \quad (9)$$
$$E(P_y^D)(y = y_D) = x - \epsilon/2. \quad (10)$$

Thus, $y^* = y_C$.

(4) **Given** $x = 1/2$ Note that $y_C = 1 - x + \epsilon$ and $y_D = x - \epsilon$. We have

$$E(P_y^C)(y = y_C) = 1/2 - \epsilon/2 \quad (11)$$
$$E(P_y^D)(y = y_D) = 1/2 - \epsilon/2. \quad (12)$$

Thus, y^* is either y_C or y_D.

(5) **Given** $1/2 < x < 2/3$ Note that $y_C = x + \epsilon$, $y_E = 1 - x + \epsilon$ and $y_D = 1 - x - \epsilon$. We have

$$E(P_y^C)(y = y_C) = 1 - x - \epsilon/2 \quad (13)$$
$$E(P_y^E)(y = y_E) = x - 1/2 - \epsilon/2 \quad (14)$$
$$E(P_y^D)(y = y_D) = 1/2 - \epsilon/2. \quad (15)$$

Thus, $y^* = y_D$.

(5) **Given** $2/3 \le x < 3/4$ Note that $y_C = x + \epsilon$, $y_E = 1 - x + \epsilon$, $y_D = 1 - x - \epsilon$ and $y_F = 3x - 2 - \epsilon$. We have

$$E(P_y^C)(y = y_C) = 1 - x - \epsilon/2 \quad (16)$$
$$E(P_y^E)(y = y_E) = x - 1/2 - \epsilon/2 \quad (17)$$
$$E(P_y^D)(y = y_D) = 1/2 - \epsilon/2 \quad (18)$$
$$E(P_y^F)(y = y_C) = 5x/2 - 3/2 - 3\epsilon/4. \quad (19)$$

Thus, $y^* = y_D$.

(6) **Given** $3/4 \le x \le 1$ Note that $y_C = x + \epsilon$, $y_E = x/3 + \epsilon$ and $y_F = x/3 - \epsilon$. We have

$$E(P_y^C)(y = y_C) = 1 - x - \epsilon/2 \quad (20)$$
$$E(P_y^E)(y = y_E) = x/3 - \epsilon/2 \quad (21)$$
$$E(P_y^F)(y = y_F) = x/2 - 3\epsilon/4. \quad (22)$$

Thus, $y^* = y_F$.

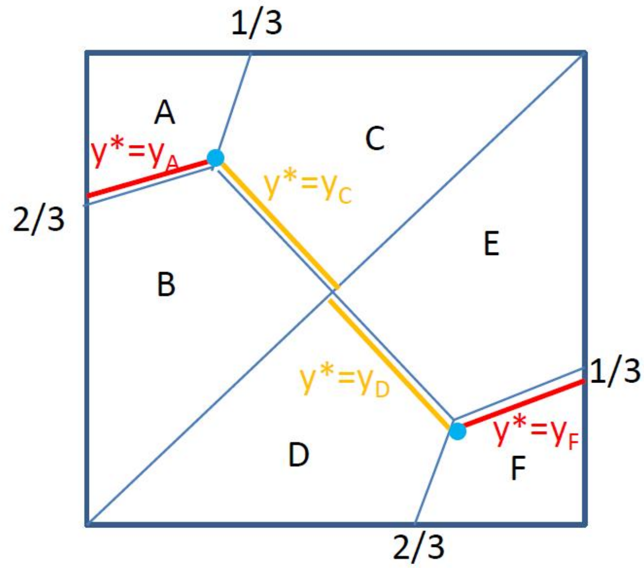
Summarizing the above results, we have the following lemma regarding the best response of Y.

Lemma 3. *The best response of Y given x, $y^*(x|z^*(x,y))$, is*

$$\begin{array}{rll} y_A = (x+2)/3 + \epsilon & \textit{for} & 0 \le x \le 1/4 \\ y_C = 1 - x + \epsilon & \textit{for} & 1/4 < x < 1/2 \\ y_C = 1/2 + \epsilon \textit{ or } y_D = 1/2 - \epsilon & \textit{for} & x = 1/2 \\ y_D = 1 - x - \epsilon & \textit{for} & 1/2 < x < 3/4 \\ y_F = x/3 - \epsilon & \textit{for} & 3/4 \le x \le 1. \end{array}$$

The lemma is described in Figure 6.

Figure 6: Best Response of Y Given x and Knowing z*



- **Best Response of X Knowing $y^*(x|z^*(x,y))$ and $z^*(x,y)$**

  Given the best responses of Y and Z, the expected payoff for player X for the choice of x is

$$\begin{aligned} E(P_x) &= 5x/6 + 1/6 + \epsilon/4 \quad \text{for} \quad 0 \le x \le 1/4 && (23)\\ E(P_x) &= -x + 1/2 + \epsilon/2 \quad \text{for} \quad 1/4 < x < 1/2 && (24)\\ E(P_x) &= \epsilon/2 \quad \text{for} \quad x = 1/2 && (25)\\ E(P_x) &= x - 1/2 + \epsilon/2 \quad \text{for} \quad 1/2 < x < 3/4 && (26)\\ E(P_x) &= -5x/6 + 1 + \epsilon/4 \quad \text{for} \quad 3/4 \le x \le 1. && (27)\\ & && (28) \end{aligned}$$

  Therefore, the best response of X (x^*) is either 1/4 or 3/4.

Proposition 4. *The following two constitute the subgame perfect equilibria in the given sequential game:*

$$x^* = 1/4, \quad y* = 3/4 + \epsilon, \quad z^* \in (1/4, 3/4 + \epsilon)$$
$$x^* = 3/4, \quad y* = 1/4 - \epsilon, \quad z^* \in (1/4 - \epsilon, 3/4).$$

The proposition is described in Figure 7. Note that there is a first mover advantage of X against Y as much as ϵ, because $E(P_x) = 3/8 + \epsilon/4$ and $E(P_y) = 3/8 - 3\epsilon/4$ at an equilibrium.

Figure 7: Best Response of X Knowing y* and z*