Choice Anomalies, Search, and Revealed Preference[*]

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Abstract

We develop a revealed preference approach to determine whether incomplete information and search can explain apparent violations of utility maximization. Since standard choice data is insufficient for this purpose, we consider non-standard data on the evolution of choice with contemplation time. We use this data to characterize a general model of sequential search. We characterize also a model of search based on standard reservation utility stopping rules. We outline an experimental protocol to elicit the required data.

Key Words: Revealed preference, search, incomplete information, revealed preference, framing effects, status quo bias, bounded rationality, stochastic choice, decision time

1 Introduction

Decision makers may not always pick the best available alternative when choice sets are large or complicated. The resulting "mistakes" have received little systematic attention within decision theory. Yet failure to pick the best available option is commonplace in the world of imperfect information. Specifically, the observation that purchasers often pay more than the minimum necessary for a good due to incomplete information lies at the heart of search theory [Stigler 1961].

We develop in this paper a decision theoretic approach to determine whether or not incomplete information and search can explain various apparently mistaken decisions. We characterize

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models of choice which incorporate the process of information search. We first consider a model of "alternative-based" search (ABS) in which a decision maker (DM) searches sequentially through the available options, choosing at all times the best thus far identified according to a fixed utility function.[1] This form of search behavior is standard in the fields of labor and macroeconomics [e.g. McCall, 1970]. We then extend the model to include a reservation-based stopping rule, under which a DM who searches until an object is identified with utility above a fixed reservation level. This reservation based search (RBS) model characterizes satisficing behavior of the form introduced by Simon [1955]. Caplin, Dean, and Martin [2009] show also that it is the optimal stopping rule for a DM who faces a fixed cost for each object searched.

While ABS and RBS represent important classes of search behavior, neither provides testable restrictions for standard choice data. Without additional ad hoc assumptions, any pattern of final choice is rationalizable with either model. We therefore consider a richer data set, which we call "choice process" data, on which to test the models. These data convey not only the final option that the DM selects, but also how their choice changes during the period of contemplation prior to making the final selection. Such a data tape has previously been considered by Campbell [1978]. It matches the criterion set forth by Block and Marshak [1960] in their pioneering model of stochastic choice, in that it is very closely related to standard choice data.[2] By so enriching the data we are able to characterize whether or not standard forms of search, which have no testable implications for final choice alone, can explain apparent violations of utility maximization.

In sections 2 and 3 we characterize ABS and RBS in the deterministic case. The key to the representations is understanding what type of behavior implies a revealed preference in the context of each model. In neither case does final choice of x over y necessarily indicate that x is preferred to y, as the decision maker may simply be unaware of y. However, in both cases, a DM who at some point chooses y and then replaces it with x is interpreted as preferring x to y. The necessary and sufficient condition for the ABS model to hold is that this information must be "consistent",

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[1] Research of Reutskaja et al. [2008] and Payne, Bettman and Johnson [1988, 1993] identifies particular circumstances in which ABS appears to be a good description of search.

[2] Much of the previous work that has examined the role of information search has made use of data that is less closely related to choice, such as the time taken in arriving at a decision [Busermeyer and Townsend, 1992; Rustichin, 2008]; direct observation of the order of information search using Mouselab [Payne, Bettman and Johnson, 1993; Ho, Camerer, and Weigelt, 1998; Johnson et al., 2002; Gabaix et al., 2006]; eye movements [Wang, Spezio and Camerer, 2006]; and verbal responses [Ericsson and Simon, 1984]. The connection of these data tapes with standard theories of choice has yet to be characterized.

in the sense of being acyclic. Under the RBS model, there may be additional revealed preference information in the final choice itself; in a set comprising objects all of which are below reservation utility, search must be complete. In such cases final choice is revealing of preference, just as in the standard model of choice.

The ABS and RBS models both treat search order as unobservable, and characterize the extent to which it is recoverable from choice process data. This makes it natural to develop stochastic versions of the model, given that search order is not a priori fixed and that there is no reason to believe that search from a given set will always take place in the same order. The stochastic version of the model is developed in section 4. While stochasticity adds to the technical intricacy of the model, there is no conceptual difference between the deterministic and the stochastic cases: the stochastic results are precise analogs of their deterministic counterparts. Section 5 applies the stochastic RBS model to capture choice anomalies, such as status quo bias, stochasticity in choice, and framing effects.

Section 6 outlines the experimental interface developed by Caplin, Dean, and Martin [2009] to gather information on the choice process and presents first findings. The results we outline confirm that many apparent mistakes result from incomplete search, and that the models developed herein help classify decision making environments according to how and how completely they will be searched. In particular, the RBS framework allows us to make rich inferences about the impact of environmental factors on the extent of the mistakes.[3]

In theoretical terms, our work is closely related to recent papers that axiomatically capture aspects of search. Salant and Rubinstein [2006] study choices made from sets presented in "list" order, effectively making the order of search observable. By contrast, we treat search order as unobservable, and characterize the extent to which it can be inferred from choice process data. Masatlioglu and Nakajima [2009] characterize choices that result from iterative search of "consideration sets" related to each alternative. Unlike this paper, they focus on how final choice is related to an initial (externally observable) reference point.[4]

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[3] Our experimental approach to observing search is related to that of Payne, Bettman and Johnson [1993] and Gabaix and Laibson [2000, 2006]. These papers use Mouselab software in order to test various models of the information gathering process. Being more standard, our choice process data set is more closely amenable to choice theoretic analysis.

[4] Consideration sets have also been extensively studied in the marketing literature {Alba and Chattopadhyay, 1985; Roberts and Lattin, 1991]. Within economics, other papers that have taken an axiomatic approach to consideration

2 ABS: The Deterministic Case

2.1 The Choice Process

Rather than recording only the alternative that is finally chosen by the DM, choice process data tracks how their choice evolves with contemplation time. As such, choice process data comes in the form of sequences of observed choices. Let X be a nonempty finite set of elements representing possible alternatives, with $\mathcal{X}$ denoting non-empty subsets of X. Let $\mathcal{Z}$ be the set of all infinite sequences from $\mathcal{X}$ with generic element $Z = \{Z_t\}_1^\infty$ with $Z_t \in \mathcal{X}$ all $t \geq 1$. For $A \in \mathcal{X}$, define $Z \in \mathcal{Z}_A \subset \mathcal{Z}$ iff $Z_t \subset A$ all $t \geq 1$.

Definition 1 *A (deterministic)* ***choice process*** (X, C) *comprises a finite set* X *and a function,* $C : \mathcal{X} \to \mathcal{Z}$ *such that* $C(A) \in \mathcal{Z}_A \; \forall \; A \in \mathcal{X}$*.*

Given $A \in \mathcal{X}$, choice process data assigns not just final choices (a subset of A), but a sequence of such choices, representing the DM's choices after considering the problem for different lengths of time. We let C_A denote $C(A)$ and $C_A(t) \in A$ denote the t-th element in the sequence C_A, with $C_A(t)$ referring to the objects chosen after contemplating A for t periods.

Choice process data represents a relatively small departure from standard choice data, in the sense that all observations represent choices, albeit constrained by time. Moreover choice process data is observable in the experimental laboratory, as discussed in section 6. We therefore see this approach as complementary to other attempts to use novel data to understand information search based on eye tracking or Mouselab [e.g. Payne, Bettman and Johnson, 1993; Gabaix et al., 2006, Reutskaja et al. 2008]. These approaches make aspects of the search process observable, yet do not connect these intermediate acts of search with their implications for choice. Choice process data misses out on potentially relevant visual and other cues on search behavior, but captures the moment at which the search that has been undertaken changes the DM's assessment of the best option thus far encountered.

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sets and search costs include Ergin [2003], Manzini and Marrioti [2007] and Ergin and Sarver [2009].

2.2 ABS

Our first model captures the process of sequential search with recall, in which the DM evaluates over time an ever-expanding set of objects, choosing at all times the best object thus far identified. Choice process data has an alternative-based search (ABS) representation if there exists a utility function and a non-decreasing search correspondence for each choice set such that what is chosen at any time is utility-maximizing in the corresponding searched set.

Definition 2 *A choice process* (X, C) *has an* ***ABS*** *representation* (u, S) *if there exists a utility function* $u : X \to \mathbb{R}$ *and a search correspondence* $S : \mathcal{X} \to \mathcal{Z}^{ND}$*, with* $S_A \in \mathcal{Z}_A$ *all* $A \in \mathcal{X}$*, such that,*

$$C_A(t) = \arg \max_{x \in S_A(t)} u(x),$$

where $\mathcal{Z}^{ND} \subset \mathcal{Z}$ *comprises non-decreasing sequences of sets in* $\mathcal{X}$*, such that* $Z_t \subset Z_{t+1}$ *all* $t \geq 1$*.*

The ABS model describes a DM who always chooses the highest utility element of the objects that they have searched. Since the DM is assumed to recall all past searches, $S_A(t)$ is non-decreasing and the choice made by the DM weakly improves over time. It is this assumption that gives the concept of ABS empirical traction. Note that the ABS model makes no assumptions concerning how or why a decision maker decides to stop searching - there is no restriction on how the function S behaves in the limit.

Given that final choice of x over y is unrevealing with incomplete search, the ABS characterization relies on an enriched notion of revealed preference. To understand the required enrichment, we consider behavioral patterns that contradict ABS. In doing this, we use the notation $C(A) = B_1; B_2; ...; B_n!$ with $B_i \in \mathcal{X} \cap A$ to indicate that the sets $B_1..B_n$ are chosen sequentially from A, with B_n being the final choice. We can readily identify four patterns of choice process data that contradict ABS.

$$\begin{aligned}
C^{\alpha}(\{x, y\}) &= x; y; x! \\
C^{\beta}(\{x, y\}) &= x; \{x, y\}; y! \\
C^{\gamma}(\{x, y\}) &= y; x!;\ C^{\gamma}(\{x, y, z\}) = x; y! \\
C^{\delta}(\{x, y\}) &= y; x!;\ C^{\delta}(\{y, z\}) = z; y!;\ C^{\delta}(\{x, z\}) = x; z!
\end{aligned}$$

C^α contains a preference reversal: the DM first switches to y from x, suggesting that x is preferred to y. However, the DM then switches back to y, indicating that y is preferred to x.C^β involves y first being revealed indifferent to x, as x and y are chosen at the same time.Yet later y is revealed to be strictly preferred to x as x is dropped from the choice set.. In C^γ the direction in which preference is revealed as between y and x changes between the two element and three element choice set. C^δ involves an indirect cycle, with separate two element sets revealing x as preferred to y, y as preferred to z, and z as preferred to x.

As these examples suggest, the appropriate notion of strict revealed preference in the case of ABS is based on the notion of alternatives being replaced in the choice sequence over time. A DM who switches from choosing y to choosing x at some later time is interpreted by the ABS model as preferring x to y. Similarly, if we ever see x and y being chosen at the same time, it must be that the DM is indifferent between the two alternatives. Hence we capture the revealed preference information implied by the ABS model in the following binary relations.

Definition 3 *Given choice process (X, C), the symmetric binary relation $\sim$ on X is defined by $x \sim y$ if there exists $A \in \mathcal{X}$ such that $\{x, y\} \subset C_A(t)$ some $t \geq 1$. The binary relation $\succ^C$ on X is defined by $x \succ^C y$ if there exists $A \in \mathcal{X}$ and $s, t \geq 1$ such that $y \in C_A(s)$, $x \in C_A(s+t)$ but $y \notin C_A(s+t)$.*

For a choice process to have an ABS representation it is necessary and sufficient for the revealed preference information captured in $\succ^C$ and $\sim$ to be consistent with an underlying utility ordering. Our characterization of ABS therefore makes use of Lemma 1, a standard result which captures the conditions under which an incomplete binary relation can be thought of as reflecting some underlying complete pre-order. Essentially, we require the revealed preference information to be acyclic.

Lemma 1 *Let P and I be binary relations on a finite set X, with I symmetric, and define PI on X as $P \cup I$. There exists a function $v : X \to \mathbb{R}$ that **respects** P and I:*

$$\begin{aligned} xPy &\implies v(x) > v(y); \\ xIy &\implies v(x) = v(y); \end{aligned}$$

*if and only if P and I satisfy **OWC** (only weak cycles): given $x_1, x_2, x_3, .., x_n \in X$ with $x = x_1 PI x_2 PI x_3 .. PI x_n = x$, there is no k with $x_k P x_{k+1}$.*

Armed with this result, we can readily establish that the key to existence of an ABS representation is for $\succ^C$ and $\sim$ to satisfy OWC.

Theorem 1 *A Choice process (X, C) has an ABS representation iff $\succ^C$ and $\sim$ satisfy OWC.*

Proof. By lemma 1, the result is equivalent to establishing that (X, C) admits an ABS representation if and only if there exists a function $v : X \to \mathbb{R}$ that respects $\succ^C$ and $\sim$ in the sense of the lemma. Certainly, if an ABS representation (u, S) exists, $x \sim y$ implies $u(x) = u(y)$ since both achieve the same maximum, while if $x \succ^C y$, then $u(x) > u(y)$ follows from $y \in C_A(s) \subset S_A(s) \subset S_A(s+t)$ with $t \geq 1$ in which $u(x)$ is maximal, while $u(y)$ is not. Conversely, if a function $v : X \to \mathbb{R}$ exists that respects $\succ^C$ and $\sim$ on X, we can define the expanding correspondence $S^* : \mathcal{X} \times \mathbb{N} \to \mathcal{X}$ by,

$$S^*_A(t) = \cup_{s \leq t} C_A(s).$$

To show that (v, S^*) form an ABS representation of (X, C), we show that $C_A(t)$ comprises all elements maximal in $S^*_A(t)$ according to $v : X \to \mathbb{R}$. Note that if $x \in C_A(t)$, then $x \succ^C y$ or $x \sim y$ all $y \in S^*_A(t)$, whereupon $v(x) \geq v(y)$ follows from the fact that v respects $\succ^C$ and $\sim$ on X. Conversely, suppose that we can find $x \in S^*_A(t)$ satisfying $v(x) \geq v(y)$ all $y \in S^*_A(t)$ but with $x \notin C_A(t)$. In this case, all $y \in C_A(t)$ satisfy $y \succ^C x$, implying that $v(y) > v(x)$, which contradiction completes the proof. ■

This condition is closely related to the standard strong axiom of revealed preference. It is readily testable, and various metrics have been developed to measure how close a data set is to satisfying such conditions (see Dean and Martin [2009] for a review).

Note that the more switches there are between objects in the choice process, the more restricted is the set of utility functions that can form part of an ABS representation. The set of equivalent representations of a choice process for which $\succ^C$ and $\sim$ satisfy OWC involve the utility function v respecting $\succ^C$ and $\sim$ on X, and the search correspondence S including at least all objects which have been chosen from all sets A at times $s \leq t$, with permissible additional elements that have utility is strictly below that associated with chosen objects according to v.

3 RBS: The Deterministic Case

Since the ABS model says nothing about the stopping rule for search, we augment it with a simple "reservation utility" stopping rule in which search continues until an object is found which has utility above some fixed reservation level, whereupon it immediately ceases.[5] The key to the empirical content of this stopping rule is that one can make inferences as to objects that must have been searched even if they are never chosen. Specifically, in any set in which the final choice has below reservation utility, it must be the case that all objects in the set are searched. Hence final choices may contain revealed preference information.

The RBS model embodies the concept of satisficing first introduced by Simon [1955]. It describes a DM who continues searching until a satisfactory object is uncovered. Such reservation-based search strategies are also standard in the macroeconomics and labor fields. Furthermore, Caplin, Dean and Martin [2009] show that a fixed reservation utility strategy is optimal in an environment in which DMs face a fixed per-object search cost, and in which the utility of objects is drawn from a fixed distribution.

Intuitively, an RBS representation is an ABS representation (u, S) in which a reservation level of utility ρ exists, and in which the above-reservation set $X^{\rho} = \{x \in X | u(x) \geq \rho\}$ plays an important role in the search process. Specifically, search stops if and only if an above-reservation item is discovered; Search is complete if there are no above-reservation items in available. In order to capture this notion formally, we define $C_A^L = \lim_{t \to \infty} C_A(t)$, as the final choice the DM makes from a set $A \in \mathcal{X}$ as well as limit search sets $S_A^L \equiv \lim_{t \to \infty} S_A(t) \in \mathcal{X}$. Note that, for finite X, the existence of an ABS representation guarantees that such limits are well defined.

Definition 4 *Choice process (X, C) has a **reservation-based search (RBS)** representation if there exists an ABS representation (u, S) and a reservation utility $\rho \in \mathbb{R}$ such that, given $A \in \mathcal{X}$,*

R1 *If $A \cap X^{\rho} = \emptyset$, then $S_A^L = A$.*

R2 *If $A \cap X^{\rho} \neq \emptyset$, then:*

(a) *there exists $t \geq 1$ such that $C_A(t) \cap X^{\rho} \neq \emptyset$;*

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[5] As in Tyson [2007] one can readily allow for reservation rules that condition on immediately observable features of the choice set, such as its cardinality.

(b) $C_A(t) \cap X^\rho \neq \emptyset \Longrightarrow S_A(t) = S_A(t+s)$ *all* $s \geq 0$.

Condition R1 demands that any set containing no objects above reservation utility is fully searched. Condition R2(a) demands that search must at some point uncover an element of the above-reservation set if present in the feasible set. Condition R2(b) states that search stops as soon as reservation utility is achieved.

As with the ABS model, the key to characterizing the RBS model is to understand the corresponding notion of revealed preference. As RBS is a refinement of ABS, it must be the case that behavior that implies a revealed preference under ABS also does so under RBS. However, the RBS model implies that some revealed preference information may also come from final choice, with sets that contain only below-reservation utility objects being completely searched.

The following cases that satisfy ABS but not RBS illustrate behaviors that must be ruled out:

$$\begin{aligned} C^\alpha(\{x,y\}) &= x; y!;\ C^\alpha(\{x,z\}) = x!;\ C^\alpha(\{y,z\}) = z! \\ C^\beta(\{x,y\}) &= x; y!;\ C^\beta(\{x,y,z\}) = x! \end{aligned}$$

In the first case, the fact that x was replaced by y in $\{x,y\}$ reveals the latter to be preferred and the former to be below reservation utility. Hence the fact that x was chosen from $\{x,z\}$ reveals z to have been searched and rejected as worse than x, making its choice from $\{y,z\}$ contradictory. In the second, the fact that x is followed by y in the choice process from $\{x,y\}$ reveals y to be preferred to x, and x to have utility below the reservation level (otherwise search must stop as soon as x is found). The limit choice of x from $\{x,y,z\}$ therefore indicates that there must be no objects of above-reservation utility in the set. However, this in turn implies that the set must be fully searched in the limit, which is contradicted by the fact that we know y is preferred to x and yet x is chosen.

In terms of ensuring an RBS representation, the critical question is how to identify all objects that are revealed as having below-reservation levels of utility. As in the above cases, we know that an object must have utility below the reservation level if we see a DM continue to search even after they have found that object. We call such an object non-terminal. Furthermore, we know that an object must be below reservation utility if, in some choice set, a directly non-terminal element is finally chosen instead of that object. We describe the union of this class of object and the non-terminal objects as indirectly non-terminal.

Definition 5 *Given choice process* (X, C) *define the non-terminal set* $X^N \subset X$ *and the indirectly non-terminal set* $X^{IN} \subset X$ *as follows,*

$$\begin{aligned} X^N &= \{x \in X | \exists A \in \mathcal{X} \text{ s.t. } x \in C_A(t) \text{ and } C_A(t) \neq C_A(t+s) \text{ some } s, t \geq 1\ \}; \\ X^{IN} &= X^N \cup \{x \in X | \exists A \in \mathcal{X},\ y \in X^N \text{ with } x, y \in A \text{ and } y \in C_A^L\}. \end{aligned}$$

Under an RBS representation, final choices in sets with below reservation utility objects contain revealed preference information: when choice is made from two objects $x, y \in X$ either of which is indirectly non-terminal, then we can conclude that the chosen object is preferred. To see this, suppose that y is indirectly non-terminal, hence has below reservation utility. In this case if it is chosen over x it must be that x was searched and rejected. Conversely, suppose that x is chosen over y. In this case either x is above reservation, in which case it is strictly preferred to y, or it is below reservation, in which case we know that the entire set has been searched, again revealing x superior. This motivates the introduction of the binary relation $\succ^L$ on X which gets united with the information from $\succ^C$ to produce the new binary relation $\succ^R$ relevant to the RBS case.

Definition 6 *Given choice process* (X, C)*, the binary relation* $\succ^L$*on* X *is defined by* $x \succ^L y$ *if* $\{x \cup y\} \cap X^{IN} \neq \emptyset$*, and there exists* $A \in \mathcal{X}$ *with* $x, y \in A$*,* $x \in C_A^L$*, yet* $y \notin C_A^L$*. The binary relation* $\succ^R$ *is defined as* $\succ^L \cup \succ^C$*.*

The behavioral condition that is equivalent to the RBS model is that the revealed preference information obtained from $\succ^R$ and $\sim$ is consistent with an underlying utility function.

Theorem 2 *Choice process* (X, C) *has an RBS representation iff* $\succ^R$ *and* $\sim$ *satisfy OWC.*

We prove theorem 2 in appendix 1, in the process characterizing the class of all reservation sets consistent with RBS, which hinges on identifying other possible below reservation sets and related preference orderings. Having characterized all sets that can form the basis of an ABS representation, the remaining forms of equivalent representation are standard. Search correspondences may include any objects that are of no strictly lower utility than those chosen at any given time. Distinct utility functions can be employed to generate a given reservation set provided both respect the associated strict preference and indifference relations.

4 The Stochastic Model

The ABS and RBS models both treat search order as unobservable, and characterize the extent to which it is recoverable from choice process data. This makes it natural to develop stochastic variants, since there is no reason to believe that search from a given set will always take place in the same order. We therefore generalize the deterministic models of section 2 and 3 to allow for stochastic choice process behavior. We do this by allowing each choice set to map onto a probability distribution over sequences of chosen objects, rather than the single sequence that we have dealt with so far. This allows us to develop stochastic versions of the RBS and ABS models, in which choice is generated from the maximization of a fixed utility function against a stochastic search sequence. The stochastic model plays an essential role in the ability of the RBS and ABS model to capture anomalous choice behavior.

4.1 ABS

In order to describe stochastic choice process data, we introduce a probability space on $\mathcal{Z}$, the class of infinite sequences from $\mathcal{X}$. This probability model is built upon standard foundations using cylinder sets.

Definition 7 *Given* $T \geq 1$ *and* $\mathcal{Y} \subset \mathcal{X}^T$, *define the cylinder set* $H(\mathcal{Y}, T)$ *by,*

$$H(\mathcal{Y}, T) = \{Z \in \mathcal{Z} |\ (Z_1, ...Z_T) \in \mathcal{Y}\}.$$

Define the algebra $\mathcal{G} = \cup_{T=1}^{\infty} \left\{ \cup_{\mathcal{Y} \subset \mathcal{X}^T} H(\mathcal{Y}, T) \right\} \in 2^{\mathcal{Z}}$, *define* $\mathcal{F} = \sigma(\mathcal{G})$ *as the* σ*-algebra generated by* $\mathcal{G}$, *and define* $\mathcal{P}$ *as all probability measures on* $(\mathcal{Z}, \mathcal{F})$, *with generic element* $P \in \mathcal{P}$.

We define the stochastic choice process as a mapping from sets $A \in \mathcal{X}$ to probability distributions over $\mathcal{Z}_A \subset \mathcal{Z}$.

Definition 8 *A* ***stochastic choice process*** $(X, \tilde{C})$ *comprises a finite set* X *and a function* $\tilde{C} : \mathcal{X} \rightarrow \mathcal{P}$ *such that* $\tilde{C}_A \equiv \tilde{C}(A)$ *has support* $\mathcal{Z}_A \subset \mathcal{Z}$.

As for the deterministic case, a stochastic choice process has an ABS representation if it can be viewed as resulting from maximization of a utility function in the context of some process of search, with the searched set never shrinking. However we allow the search process to be stochastic. We will use $\tilde{S} : \mathcal{X} \to \mathcal{P}^{ND}$ to denote a stochastic search function, where: $\mathcal{P}^{ND} \subset \mathcal{P}$ identify probability measures on $(\mathcal{Z}, \mathcal{F})$ with support $\mathcal{Z}^{ND}$, the non-decreasing elements of $\mathcal{Z}$. Given $A \in \mathcal{X}$ and $F \in \mathcal{F}$, let $\tilde{C}_A(F)$, $\tilde{S}_A(F)$ respectively denote the measure assigned to F by $\tilde{C}(A)$, $\tilde{S}(A)$.[6]

Definition 9 *A stochastic choice process* $(X, \tilde{C})$ *has a* ***stochastic ABS*** *representation* $(u, \tilde{S})$ *if there exists* $u : X \to \mathbb{R}$ *and* $\tilde{S} : \mathcal{X} \to \mathcal{P}^{ND}$ *such that* $\tilde{C}$ *is the stochastic choice process derived by optimizing* u *against* $\tilde{S}$,

$$\bar{C}_A(F) = \tilde{S}_A\left(\left\{Z \in \mathcal{Z} \middle| \left\{\arg \max_{x \in Z_t} u(x)\right\}_{t=1}^{\infty} \in F\right\}\right), \text{ all } A \in \mathcal{X}, F \in \mathcal{F}.$$

The theorem that characterizes the stochastic ABS representation is essentially identical to that in the deterministic case. It simplifies notation to define join and replacement sets $J^{xy}, R^{xy} \subset \mathcal{Z}$ for $x, y \in X$, where J^{xy} is the set of choice processes in which x and y are chosen at the same time, while R^{xy} are those in which y is replaced by x.

$$\begin{aligned} J^{xy} &= \{Z \in \mathcal{Z} | \{x, y\} \subset Z_t \text{ some } t \geq 1\}; \\ R^{xy} &= \{Z \in \mathcal{Z} | y \in Z_s,\ x \in Z_{s+t},\ y \notin Z_{s+t} \text{ some } s, t \geq 1\}; \end{aligned}$$

Measurability of $J^{xy}, R^{xy} \subset \mathcal{Z}$ is established in appendix 2.

For purposes of establishing the stochastic ABS representation, we define x to be revealed strictly preferred to y if R^{xy} has strictly positive measure, and x to be revealed indifferent to y if the set J^{xy} has strictly positive measure.

Definition 10 *Given a stochastic choice process* $(X, \tilde{C})$, *the binary relation* $\sim^{\tilde{C}}$ *on* X *is defined by* $x \sim^{\tilde{C}} y$ *if there exists* $A \in \mathcal{X}$ *with* $x, y \in A$ *and* $\tilde{C}_A(J^{xy}) > 0$. *The binary relation* $\succ^{\tilde{C}}$ *on* X *is defined by* $x \succ^{C} y$ *if there exists* $A \in \mathcal{X}$ *with* $x, y \in A$ *and* $\tilde{C}_A(R^{xy}) > 0$.

As before, the condition for the characterization is that this revealed preference information is consistent with a fixed underlying utility function.

Theorem 3 *A stochastic choice process* $(X, \tilde{C})$ *has a stochastic ABS representation* $(u, \tilde{S})$ *iff* $\succ^{\tilde{C}}$ *and* $\sim^{\tilde{C}}$ *satisfy OWC.*

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[6] That the set of $Z \in \mathcal{Z}$ with arg $\max_{x \in Z_t} u(x)_{t=1}^{\infty} \in F$ is measurable is demonstrated in appendix 2.

4.2 RBS

As in the deterministic case, the definition of a stochastic RBS representation requires the analysis of limit behavior. Given $B \in \mathcal{X}$, we define L^B to be the $\mathcal{F}$-measurable subset of $\mathcal{Z}$ with limit B,

$$L^B = \left\{ Z \in \mathcal{Z} | \lim_{t \to \infty} Z_t = B \right\}.$$

In appendix 2 it is shown that a stochastic choice process model $(X, \tilde{C})$ with stochastic ABS representation $(u, \tilde{S})$ necessarily assigns full measure to the set in which limits exist,

$$\tilde{C}_A \left\{ \cup_{B \in \mathcal{X}} L^B \right\} = 1.$$

Hence, given a stochastic choice process model $(X, \tilde{C})$ with stochastic ABS representation $(u, \tilde{S})$ and $A \in \mathcal{X}$, we can define limit choice and search probability measures $\tilde{C}^L_A, \tilde{S}^L_A$ on $\mathcal{X}$ endowed with the discrete sigma-algebra,

$$\tilde{C}^L_A(B) = \tilde{C}_A(L^B) \text{ and } \tilde{S}^L_A(B) = \tilde{S}_A(L^B) \text{ any } B \in \mathcal{X}.$$

As in the deterministic case, the definition of stochastic RBS involves a utility function $u : X \to \mathbb{R}$ and a level of reservation utility ρ which together identify above reservation set $X^\rho \equiv \{x \in X | u(x) \geq \rho\}$. Given $Z \in \mathcal{Z}$, a key random variable in the stochastic RBS representation is the first time that reservation utility is hit. To simplify notation in the stochastic version of RBS, we let $H^\rho_u : \mathcal{Z} \longrightarrow \mathbb{N} \cup \infty$ denote this first hitting time associated with utility function u and reservation utility level ρ,

$$H^\rho_u(Z) = \begin{cases} \inf_{t \geq 1} \{Z_t \cap X^\rho\} \neq \emptyset, \text{ if } \{Z_t \cap X^\rho\} \neq \emptyset \text{ some } t; \\ \infty \text{ otherwise.} \end{cases}$$

That hitting times are $\mathcal{F}$-measurable functions is standard.

We use the notion of hitting times to define the stochastic version of the RBS model.

Definition 11 *A stochastic choice process $(X, \tilde{C})$ with stochastic ABS representation $(u, \tilde{S})$ has a **stochastic RBS** representation if there exists $\rho \in \mathbb{R}$ such that, given $A \in \mathcal{X}$,*

RS1 *If $A \cap X^\rho = \emptyset$, then $\tilde{S}^L_A(A) = 1$*

RS2 *If $A \cap X^\rho \neq \emptyset$, then:*

(a) $\tilde{S}_A \{Z \in \mathcal{Z} | H_u^\rho(Z) \text{ is finite}\} = 1$;

(b) $\tilde{S}_A \left\{Z \in \mathcal{Z} | \tilde{S}_A^L = \tilde{S}_A\left(H_u^\rho(Z)\right)\right\} = 1$.

As with ABS, the stochastic RBS characterization is the precise analog of the deterministic version, and relies on the identification of directly and indirectly non-terminal sets. We define $\Delta^y \subset \mathcal{Z}$ to be the set of sequences in which $y \in X$ appears at some point, but the sequence changes thereafter. Measurability is established in appendix 2.

Definition 12 *Given a stochastic choice process $(X, \tilde{C})$, define the* ***non-terminal set*** $\tilde{X}^N \subset X$ *as,*

$$\tilde{X}^N = \left\{x \in X | \exists A \in \mathcal{X} \text{ with } x \in A \text{ and } \tilde{C}_A(\Delta^x) > 0\right\}.$$

Define the ***indirectly non-terminal*** *set $\tilde{X}^{IN}$ as $\tilde{X}^N$ and elements rejected with positive probability in favor of an element of X^N,*

$$\tilde{X}^{IN} = \tilde{X}^N \cup \{x \in X || \exists A \in \mathcal{X} \text{ , } y \in \tilde{X}^N \text{ with } x, y \in A \text{ and } \tilde{C}_A^L(\{y\}) > 0\}.$$

The definition of revealed preference in the stochastic RBS model can now proceed in line with the deterministic case.

Definition 13 *Given a stochastic choice process $(\tilde{X}, C)$, the binary relation $\succ^{\tilde{L}}$ on X is defined by $x \succ^{\tilde{L}} y$ if $\{x \cup y\} \cap \tilde{X}^{IN} \neq \emptyset$, and there exists $A \in \mathcal{X}$ with $x, y \in A$ with $\tilde{C}_A^L\{x\} > 0$ and $\tilde{C}_A(J^{xy}) = 0$. Binary relation $\succ^{\tilde{R}}$ is defined as $\succ^{\tilde{L}} \cup \succ^{\tilde{C}}$.*

Using this definition, the standard application of Lemma 1 characterizes existence of an RBS representation.

Theorem 4 *A stochastic choice process model $(X, \tilde{C})$ has a stochastic RBS representation $(u, \tilde{S}, \rho)$ if and only if $\succ^{\tilde{R}}$ and $\sim^{\tilde{C}}$ satisfy OWC.*

4.3 Sketch of Proofs

The proofs of theorem 3 and of theorem 4 are detailed in appendix 3. We limit ourselves in this discussion to presenting structural elements. Both proofs work by reducing the stochastic case to its deterministic counterpart. The key step involves showing that nothing is lost by "compressing" choice process data by removing time periods in which choice does not change.

Definition 14 *The stochastic choice process* $(X, \tilde{C})$ *is compressed if* $\tilde{C}_A(\mathcal{Z}^{COM}) = 1$ *for all* $A \in \mathcal{X}$*, where,*

$$\mathcal{Z}^{COM} \equiv \{Z \in \mathcal{Z} | Z_t = Z_{t+1} \Longrightarrow Z_t = Z_{t+s} \text{ all } s \geq 1\}.$$

In the first step of the reduction, a given stochastic choice process $(X, \tilde{C})$ is associated with a unique compressed choice process by removing all periods of constancy (see appendix 3 for details). The process of compression reduces to equivalence an infinite number of choice processes differing only in the delay between switches.

The first observation that makes compression of value is the invariance of key properties under compression and its inverse, decompression. It is immediate that the orderings $\succ^{\tilde{R}}$, $\succ^{\tilde{C}}$and $\sim^{\tilde{C}}$ are preserved under both operations. It is equally immediate that ABS and RBS survive both under compression and decompression, since one uses exactly the same utility function and reservation utility in the representation of the original process and its transformation, using compression only to change the search correspondence by removing repetition in the case of compression, and inverting suitably in the process of decompression.

The second observation that makes compression of value is that any compressed process that satisfies ABS is "finite", in that only a finite number of sequences have strictly positive probability. Conversely, any compressed stochastic choice process for which $\succ^{\tilde{C}}$ and $\sim^{\tilde{C}}$ satisfy OWC is finite. While the formal definitions and proof are in appendix 3, the intuition is simple. Both ABS and OWC imply that a compressed stochastic choice process must stop changing within a number of periods that matches the cardinality of the power set of $\mathcal{X}$.

The bottom line of this reduction process is that the proofs in of theorems 3 and 4, detailed in the appendix, are provided only for finite models, with the extension to the general case being immediate. The critical observation in establishing the finite case is that any finite stochastic choice processes $(X, \tilde{C})$ can be identified with an appropriately defined convex combinations of deterministic choice processes. The result opens the door to possible future application of the Herstein-Milnor mixture space axioms to models of the choice process.

5 RBS and Non-Standard Behavior

In this section we discuss how the stochastic RBS framework can be used to model various choice anomalies: framing effects; status quo bias; and stochastic choice.[7]

5.1 Framing Effects

A natural application of our model is to framing effects, whereby seemingly unimportant changes in the environment can lead to changes in final choice. In the case of the RBS model, this can occur if the frame impacts either the order of search or the level of reservation utility.

To model these effects, let Γ comprise abstract elements $\gamma \in \Gamma$ that we refer to as frames. These frames may represent different ways in which objects are physically displayed to the DM. Let $\Phi : \Gamma \to \bar{\mathcal{C}}$ be a mapping from frames to the class $\bar{\mathcal{C}}$ of stochastic choice processes on $(\mathcal{Z}, \mathcal{F})$, with $\Phi(\gamma)$ the process associated with $\gamma \in \Gamma$. We seek to characterize data sets in which all choice processes regardless of frame can be derived from a common underlying utility function but with frame-specific search orders and reservation utilities. Such a characterization is experimentally useful, since it indicates conditions under which one can derive information on preferences in a low search cost (hence high reservation utility) environment that will apply equally in a higher search cost (hence lower reservation utility) frame in which choice process data yields less direct evidence on preferences. It turns out that we need to apply OWC to a binary relation that appropriately unifies revealed preference information across frames. In the statement, $\bar{\mathcal{S}}$ denotes the set of all stochastic search processes on $(\mathcal{Z}, \mathcal{F})$.

Definition 15 *Define* $x \succ^{\tilde{R}(\Gamma)} y$ *if* $x \succ^{\tilde{R}} y$ *according to some stochastic choice process* $\Phi(\gamma)$ *for some* $\gamma \in \Gamma$ *. Similarly define* $x \sim^{\tilde{C}(\Gamma)} y$ *if* $x \sim^{\bar{R}} y$ *according to some stochastic choice process* $\Phi(\gamma)$ *for some* $\gamma \in \Gamma$*.*

Theorem 5 *Given finite set* X*, frames* Γ*, and* $\Phi : \Gamma \to \bar{\mathcal{C}}$*, there exists a utility function* $u : X \to \mathbb{R}$*, a family of reservation utilities* $\rho : \Gamma \to \mathbb{R}$*, and family of stochastic search processes* $\Theta : \Gamma \to \bar{\mathcal{S}}$ *such that* $(u, \Theta(\gamma), \rho(\gamma))$ *forms a stochastic RBS representation of* $\Phi(\gamma)$ $\forall\ \gamma \in \Gamma$ *iff* $\succ^{\tilde{R}(\Gamma)}$ *and* $\sim^{\tilde{C}(\Gamma)}$

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[7] In practice, we understand that information search cannot explain all instances of such phenomena and that other forces will also be at work.

satisfy OWC.

5.2 Status Quo Bias

One particular class of framing effect that can be explored using the RBS model is status quo bias - the increased likelihood of selecting a particular object simply because it is the status quo, or currently selected option [Samuelson and Zeckhauser, 1988]. We can model such behavior as a particular class of the framing model introduced above, in which each status quo gives rise to its own frame. In order to capture status quo bias, we posit that the status quo object is always the first object searched in any choice environment.

Under this assumption, the stochastic RBS model makes particular predictions about how status quo will affect choice. For above-reservation utility objects, status quo bias will be complete: when such objects are the status quo then they will always be chosen, as the DM is immediately aware of their existence and will indulge in no further search. However, if the status quo object is below reservation utility then it will not be chosen unless it is the highest utility object in the choice set, in which case it will be chosen regardless of the status quo, as the stochastic RBS model implies that search will be complete in such cases. Thus, the RBS model predicts that status quo bias will be of one of two extremes: either an object will always be chosen when it is the status quo, or status quo will have no effect.

5.3 Stochastic Choice

It is clear that the stochastic RBS model can give rise to stochastic choice in the form of a probability distribution over final choices. Even with a fixed utility function, final choice will be random if the order is search is random and search is incomplete. However this distribution will be of a particular form: choice may be stochastic among above reservation objects, while objects with below reservation utility are never chosen. In the simplest possible case with all search orders being equally probable, final choice is deterministic and consistent for choice sets made up only of below-reservation items, whereas for choice sets containing above-reservation items, there is an equal chance of choosing any such item. Observed stochasticity in choice will therefore increase as reservation utility falls.

6 Eliciting Choice Process Data in the Laboratory

The above analysis makes use of choice process data in a central and fundamental way. Hence the usefulness of these results depends on our ability to generate and observe these data. In a companion paper [Caplin, Dean and Martin 2009], we describe and implement an experimental methodology designed precisely to generate choice process data. We show that the resulting data is well described by both the ABS and RBS model.

The key to the experimental design is the mechanism by which we incentivize subjects to record their currently most preferred option in real time. To this end, we allow subjects, who are choosing among options presented to them on a computer screen, to change their selected option as many times as they like during the course of the experimental round. The option that is recorded as their actualized choice from that round is the object they had selected at some randomly determined time, which time is only revealed at the end of the round. The fact that there is always a positive probability that the current selection will be chosen provides an incentive to always select the currently most preferred option.

The resulting experimental data has several attractive properties. First, subjects make several selections in the course of a round. Thus, the data we record is richer than standard choice data. Moreover, subjects improve their choices - in general switching from lower value to higher value objects. In the context of the experiment this is equivalent to finding positive support for the ABS model of search.[8] Finally, it appears that in most cases the experimental design does not significantly impact the final choices that are made, making the results applicable to standard choice data.

As well as finding support for ABS-style search, we also find that behavior is well approximated by RBS. Moreover, we can use the resulting estimates of reservation utilities to draw conclusions about whether subjects' search strategies respond optimally to environmental cues. For example, our results suggest that reservation utility falls as the complexity of choice objects increases, in line with the optimal search strategy.

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[8] In the experimental paper we use choice objects that have a fixed underlying value that requires effort to uncover. Specifically, we use sequences of addition and subtraction operations, with the dollar value of the object equal to the sum - for example "four plus three minus two plus one" is worth \$6. The fact that the underlying value is observable dispenses with the need for the revealed preference approach adopted in this paper.

7 Concluding Remarks

Incomplete information may explain many apparent deviations from utility maximizing behavior. Standard choice data does not allow one to pin down when such deviations are caused by changing preferences, and when they result from incomplete information. We develop clean procedures for accomplishing this separation by expanding beyond standard choice data to include data on the evolution of choice with time. We characterize standard alternative-based and reservation-based procedures that are ubiquitous in search theory. Experimental investigation of choice process data is ongoing.

8 Appendix 1: RBS

We present two characterization results. The first characterizes all reservation sets in any RBS representation of a choice process model, and is the key to establishing theorem 2. The second indicates the class of search correspondences and reservation utilities consistent with such a representation for a given reservation set. The starting point is a generalization of the ordering $\succ^L$ of section 3 to $\succ^L_D$, the revealed preference relation generated by an arbitrary below-reservation set D.

Definition 16 *Given a choice process model (X, C) and set $D \in \mathcal{X}$, the binary relation $\succ^L_D$on X is defined by $x \succ^L_D y$ if $\{x \cup y\} \cap D \neq \emptyset$, and there exists $A \in \mathcal{X}$ with $x, y \in A$, $x \in C^L_A$, yet $y \notin C^L_A$. The binary relation $\succ^R_D$ is defined as $\succ^L_D \cup \succ^C$.*

Proposition 1 now characterizes the set of possible below-reservation sets that can generate an RBS representation.

Proposition 1 *A choice process model (X, C) admits an RBS representation with reservation set $X \backslash D$ if and only if:*

1. $X^N \subset D$

2. If $x \in D$ and $x \succsim^R_D y$, then $y \in D$.

3. $\succsim_D^R$ satisfies OWC.

Proof. To prove sufficiency, we note from lemma 1 that (3) implies existence of $u : X \to \mathbb{R}$ that respects $\succ_D^R$ and $\sim$ on X. Define

$$\rho = \frac{\max_{x \in D} u(x) + \min_{x \in X \backslash D} u(x)}{2}.$$

Note from (2) that $C^L\{x, y\} = y$ whenever $y \in X \backslash D$ and $x \in D$, implying $y \succ_D^R x$ and $u(y) > u(x)$ and hence that $X/D = X^\rho$. Mimicking the proof of theorem 1, one can then define a search correspondence such that (u, S) that together form an ABS representation.

$$S_A(t) = \begin{cases} \cup_{s \leq t} C_A(s) \text{ for } t < T(A); \\ \cup_{s \leq T(A)} C_A(s) \cup L(A) \text{ for } t \geq T(A); \end{cases}$$

where $T(A) \equiv \min\{t \geq 1 | C_A(t) = C_A^L\})$ is the time at which choice first achieves its limit and $L(A)$ comprises all elements of A with utility strictly below $\max_{x \in C_A^L} u(x)$. We now show that all requirements for (u, S) and ρ together to form an RBS representation with reservation set $X \backslash D$ are met:

- R1: When $A \cap X^\rho = \emptyset$, and so $A \subset D$, we know that $x \in C_A^L$, $y \notin C_A^L \implies x \succ_D^L y$, so that $u(x) > u(y)$. Hence $C_A^L = \arg\max_{\{x \in A\}} u(x)$ with $S_A^L = A$ by construction.
- R2(a): If $A \cap X^\rho \neq \emptyset$ and so $A \cap X \backslash D \neq \emptyset$, then $C_A^L \cap D = \emptyset$ since $x \in C_A^L \cap D$, $y \notin C_A^L \implies u(x) > u(y)$ contradicting the fact that utility is strictly higher on $X \backslash D$ than on D. Hence there exists $t \geq 1$ such that $C_A(t) \cap X^\rho \neq \emptyset$.
- R2(b): If $C_A(t) \cap X^\rho \neq \emptyset$, then $C_A(t) \cap X^N = \emptyset$ by (1), implying directly that $C_A(t + s) = C_A(t + s)$ all $s \geq 1$, so that $T(A) \equiv \min\{t \geq 1 | C_A(t) \cap X^\rho \neq \emptyset\})$, so that $S_A(T(A) + s) = S_A(T(A) + s)$ all $s \geq 1$ as required.

That condition (1) of the proposition is necessary for an RBS representation follows directly from property R2(b) of RBS definition, which implies that $X^N \subset D$ is required for D to be a reservation set. Given lemma 1, to prove that (3) is necessary it suffices to show that u represents $\succ_D^R$ and $\sim$ in any RBS representation (u, S, ρ), where $D = X \backslash X^\rho$ and X^ρ is the corresponding reservation set. The fact that u represents $\succ^C$ and $\sim$ is direct since (u, S) form an ABS representation of (X, C). To see that $\succ_D^L$ is respected, suppose to the contrary that $x \succ_D^L y$ but $u(y) \geq u(x)$. Note in this case that $x \in D$, since $y \in D \Longrightarrow x \in D$ and $\{x \cup y\} \cap D \neq \emptyset$ by definition of $x \succ_D^L y$. But then by R1, $x \in C_A^L \Longrightarrow C_A^L = \arg\max_{x \in A} u(x)$ hence $u(y) < u(x)$ since $y \notin C_A^L$. This contradiction establishes that u indeed represents $\succ_D^R$ and $\sim$. With this we know that condition (2) of the proposition is necessary, since $x \in D \Longrightarrow u(x) < \rho$ whereupon $x \succsim_D^R y$ implies $u(y) < \rho$, hence $y \in D$, completing the proof. ■

Proof of Theorem 2 To prove sufficiency, we show that the conditions of the proposition are satisfied in this case for $D = X^{IN}$. For (1) and (3) this is direct. Hence it suffices to establish that if $x \in X^{IN}$ and $x \succsim^R y$, then $y \in X^{IN}$. By definition $x \in X^{IN}$ implies that we can find $z \in X^N$ with $z \succeq^L x$. Now, if $C_{\{y,z\}}^L = y$, we have that $x \succsim^R y \succ^R z \succeq^L x$ violating OWC. Thus it must be the case that $z \succeq^L y$, implying by definition that $y \in X^{IN}$, as required. To show that $\succ^R$ and $\sim$ satisfying OLC is necessary for (X, C) to have any RBS representation (u, S, ρ), it suffices by lemma 1 to show that such $u : X \to \mathbb{R}$ must respect $\succ^R$ and $\sim$. This follows directly for $\succ^C$ and $\sim$ since (u, S) form an ABS representation of (X, C). To confirm that $u : X \to \mathbb{R}$ respects $\succ^L$, consider $A \in \mathcal{X}$ with $x, y \in A$, $x \in C_A^L$, $y \notin C_A^L$, and x or $y \in X^{IN}$. There are two cases.

- If $u(x) < \rho$, then $x \in C_A^L \Longrightarrow A \cap X^\rho = \emptyset$ by R2(a) hence $S_A^L = A$ by R1, hence $u(y) < u(x)$ all $y \in A$ with $y \notin C_A^L$.
- If $u(x) \geq \rho$, then $x \notin X^{IN}$ follows directly from condition 2(b) of the RBS definition, so that $y \in X^{IN} \subset X \backslash X^\rho$, and $u(y) < \rho \leq u(x)$.

9 Appendix 2: Measurability

We show that various sets are contained in the σ-algebra $\mathcal{F}$.

- $\mathcal{Z}^{COM}$ and $\mathcal{Z}^{ND}$: Given $T \geq 1$, define $\mathcal{ND}^T$ as all subsets of $\mathcal{X}^T$ that are non-diminishing, $Z_t \subset Z_{t+1}$ all $1 \leq t \leq T$, and $\mathcal{NR}^T$ as all subsets of $\mathcal{X}^T$ in which there is no immediate repetition, $Z_t \neq Z_{t+1}$ any $1 \leq t \leq T-1$, and note that,

$$
\begin{aligned}
\mathcal{Z}^{ND} &= \cap_{T=1}^{\infty} \left\{ Z \in \mathcal{Z} | \left(Z_1, .., Z_T\right) \in \mathcal{ND}^T \right\} \in \mathcal{F}; \\
\mathcal{Z}^{COM} &= \cup_{t=1}^{\infty} \left\{ \cap_{s=1}^{\infty} \left\{ Z \in \mathcal{Z} | \left(Z_1, .., Z_t\right) \in \mathcal{NR}^t, Z_t = Z_{t+s} \right\} \right\} \in \mathcal{F}.
\end{aligned}
$$

- That$\{Z \in \mathcal{Z} | \{\arg\ \ \max_{x\in Z_t} u(x)\}_{t=1}^{\infty} \in F\} \in \mathcal{F}$ for any $F \in \mathcal{F}$, note that it can be expressed as follows as a countable collection of cylinder sets,

$$\cap_{T=1}^{\infty}\left\{Z \in \mathcal{Z} | \exists\ Y \in F \text{ s.t. } \arg\ \max_{x\in Z_t} u(x) = Y_t\ \forall\ t \in \{1,...,T\}\right\}.$$

- For any $x, y \in X$, the sets J^{xy}, R^{xy}, and Δ^x. Given $A \in \mathcal{X}$, define W_A as all supersets of A and $W_A^C \subset \mathcal{X}$ as its complement. Define the cylinder sets $\mathcal{W}_A(t), \mathcal{W}_A^C(t) \in \mathcal{G}$ by,

$$\begin{aligned} \mathcal{W}_A(t) &\equiv \{Z \in \mathcal{Z}|Z_t \in W_A\}; \\ \mathcal{W}_A^C(t) &\equiv \{Z \in \mathcal{Z}|Z_t \in W_A^C\}. \end{aligned}$$

- Note that:

$$\begin{aligned} J^{xy} &= \cup_t \mathcal{W}_{\{x,y\}}(t) \in \mathcal{F}; \\ R^{xy} &= \cup_{t=1}^{\infty}\left\{\mathcal{W}_{\{y\}}(t) \cap \left\{\cup_{s=1}^{\infty}\left\{\mathcal{W}_{\{y\}}^C(t+s) \cap \mathcal{W}_{\{y\}}(t+s)\right\}\right\}\right\} \in \mathcal{F}; \\ \Delta^x &= \cup_{t=1}^{\infty}\left\{\cup_{B\in W_{\{x\}}}\left\{\cup_{s=1}^{\infty}\left\{Z \in \mathcal{Z}|Z_t = B, Z_{t+s} \neq B\right\}\right\}\right\} \in \mathcal{F}. \end{aligned}$$

- $\mathcal{Z}^{NCY} = \{Z \in \mathcal{Z}|Z_{t+1} \neq Z_t \implies Z_{t+s} \neq Z_t$ any $s \geq 1\}$: (see appendix 3). First, index all sets in $\mathcal{X}$, $A_1, , A_m, .., A_M$, with M the cardinality of $\mathcal{X}$. Define $\Pi(M)$ to be all permutations of the first $m \leq M$ integers. Given $\pi^m \in \Pi(M)$, define the countable set $\Upsilon(\pi^m)$ to comprise all strictly increasing sets of m natural numbers,

$$\Upsilon(\pi^m) = \{T^m = \{T_1^m, T_2^m, .., T_m^m)|T_1^m = 1,\ T_i^m \in \mathbb{N} \text{ and } T_i^m < T_{i+1}^m \text{ all } i \geq 1\}.$$

That $\mathcal{Z}^{NCY} \in \mathcal{F}$ follows since it is a countable union of cylinder sets,

$$\cup_{\pi^m\in\Pi(M)}\cup_{T^m\in\Upsilon(\pi^m)}\left\{Z \in \mathcal{Z}|Z_t = A_{\pi_i^m} \text{ for } T_i^m \leq t < T_{i+1}^m,\ 1 \leq i \leq m-1;\ Z_t = A_{\pi_m^m} \text{ for } t \geq T_m^m\ \right\}.$$

- $\mathcal{E}(Y)$: (see appendix 3). Given K non-negative integers s_k define $S_0 = 0$ and partial sums $S_k = \sum_{j=1}^{k} s_j$ enabling the following short definition:

$$\mathcal{E}(Y) = \cap_{K=1}^{\infty}\left\{\cup_{s_K=1}^{\infty}....\left\{\cup_{s_1=1}^{\infty}\{Z \in \mathcal{Z}|Z_\tau = Z_k \text{ for } S_{k-1}+1 \leq \tau \leq S_k \text{ and } 1 \leq k \leq K\}\right\}\right\} \in \mathcal{F}.$$

Proposition 2 *If $(X, \tilde{C})$ permits of a stochastic ABS representation $(u, \tilde{S})$, then for any $A \in \mathcal{X}$,*

$$\tilde{C}_A\left\{\cup_{B\in\mathcal{X}} L^B\right\} = 1.$$

Proof. Since $(X, \tilde{C})$ has an ABS representation $(u, \tilde{S})$, we know that $\tilde{S}_A(\mathcal{Z}^{ND}) = 1$. Note that since $\mathcal{X}$ is finite, limit elements exist for all $Z \in \mathcal{Z}^{ND}$, establishing that $\tilde{S}_A\left\{\cup_{B\in\mathcal{X}} L^B\right\} = 1$. Now note that if $Z \in \cup_{B\in\mathcal{X}} L^B$, then $\{\arg\ \max_{x\in Z_t} u(x)\}_{t=1}^{\infty} \in \cup_{B\in\mathcal{X}} L^B$, as, $Z \in \cup_{B\in\mathcal{X}} L^B$ implies that there must be some t such that $Z_t = Z_{t+s}\ \forall\ s \geq 0$, thus it must be the case that arg $\max_{x\in Z_t} u(x) =$ arg $\max_{x\in Z_t} u(x)\ \forall\ s \geq 0$. Hence,

$$\tilde{C}_A\left\{\cup_{B\in\mathcal{X}} L^B\right\} = \bar{S}\left\{Z \in \mathcal{Z} | \left\{\arg\ \max_{x\in Z_t} u(x)\right\}_{t=1}^{\infty} \in \cup_{B\in\mathcal{X}} L^B\right\} \geq \bar{S}\left\{\cup_{B\in\mathcal{X}} L^B\right\} = 1.$$

■

10 Appendix 3: Theorems 3 and 4

We first formally define compression, from which it follows immediately that it is sufficient to prove theorems 3 and 4 for compressed stochastic choice processes. We then show that compressed stochastic choice processes of interest are finite, further simplifying the requirements to establishing 3 and 4 for finite stochastic choice processes. Next, we show that finite stochastic choice processes can be represented as weighted averages of deterministic processes. We close out by proving theorems 3 and 4 for the finite case, which proof is general in light of the earlier results.

10.1 Compression

Definition 17 *Given $Z \in \mathcal{Z}$, define the set of times at which Z changes in sequential fashion starting with $\tau_1(Z) = 1$ as follows;*

$$\tau_{j+1}(Z) = \begin{cases} \min_{s\geq 1}\{Z_{\tau_j(Z)+s} \neq Z_{\tau_j(Z)}\} & \text{if } \exists s \geq 1 \text{ s.t } Z_{\tau_j(Z)+s} \neq Z_{\tau_j(Z)}; \\ \infty \text{ if } Z_{\tau_j(Z)+s} = Z_{\tau_j(Z)} \text{ all } s \geq 1. \end{cases}$$

Let $J(Z) \in \mathbb{N} \cup \infty$ be the number of distinct points of change, and define the compression of any element $Z \in \mathcal{Z}$, $D(Z) \in \mathcal{Z}^{COM}$, by removing all time indices in which there is repetition and repeating the limit element if there is any repetition,

$$D(Z) = \begin{cases} (Z_{\tau_1(Z)}, ..., Z_{\tau_j(Z)}, ..Z_{\tau_{J(Z)}(Z)}, ...Z_{\tau_{J(Z)}(Z)}, ..Z_{\tau_{J(Z)}(Z)}) \text{ if } J(Z) \text{ is finite}; \\ (Z_{\tau_1(Z)}, ..., Z_{\tau_j(Z)}, ..) \text{ if } J(Z) = \infty. \end{cases}$$

Given $Y \in \mathcal{Z}^{COM}$, define the equivalence classes of compressed elements of $\mathcal{E}(Y) \subset \mathcal{Z}$ ((the proof that $\mathcal{E}(Y) \in \mathcal{F}$ is in appendix 2),

$$\mathcal{E}(Y) = \{Z \in \mathcal{Z} | D(Z) = Y\}.$$

Given a measure $P \in \mathcal{P}$, we define its compression $D^P \in \mathcal{P}$ by shifting probabilities onto the compressed representative of each equivalence class,

$$D^P(Y) = \begin{cases} P(\mathcal{E}(Y)) \text{ for } Y \in \mathcal{Z}^{COM}; \\ 0 \text{ for } Y = \mathcal{Z} \backslash \mathcal{Z}^{COM}. \end{cases} .$$

10.2 Compression and Finiteness

Proposition 3 *A compressed SCP that has an ABS representation or for which $\succ^{\tilde{C}}$ and $\sim^{\tilde{C}}$ satisfy OWC is* ***finite****, in that there exists a finite set $G \in \mathcal{F}$ such that $\tilde{C}_A(G) = 1$ all $A \in \mathcal{X}$.*

Proof. To show that compression and ABS imply that the SCP is finite, let $M = |\mathcal{X}|$ and let $\mathcal{Z}(M) \in \mathcal{F}$ be sequences that are unchanging after period M :

$$\mathcal{Z}(M) = \{Z \in \mathcal{Z} | Z_t = Z_s \ \forall \ t, s > M\}.$$

It is intuitive that a compressed choice sequence with an ABS representation satisfies $\bar{C}_A\left(\mathcal{Z}(M)\right) = 1 \ \forall \ A \in \mathcal{X}$. To confirm, consider the union of all cylinder sets with $Z_t \neq Z_s$ some $t, s > M$. If any element Z in this set is to be in $\mathcal{Z}^{COM}$, it must be the case that, for some $r, w < s$, $Z_r = Z_w$ and $r \neq w \pm 1$. Consider now the cylinder sets defined by,

$$\{Z \in \mathcal{Z} | Z_t \neq Z_s, \ Z_r = Z_w \ \}.$$

Now take any k such that $r < k < w$. and consider the cylinder set

$$\{Z \in \mathcal{Z} | Z_t \neq Z_s, \ Z_k \neq Z_r = Z_w \ \}.$$

These cylinder sets must have measure zero in any choice process that has an ABS representation, as the set of search sequences such that

$$\arg \max_{x \in S_A(k)} u(x) \neq \arg \max_{x \in S_A(r)} u(x) = \arg \max_{x \in S_A(w)} u(x),$$

is measure zero (as any such sequence would be non-increasing). As $\mathcal{Z} \backslash \mathcal{Z}(M)$ can be obtained by the repeated countable union across $\{Z \in \mathcal{Z} | Z_t \neq Z_s, \ Z_r = Z_w \ \}$, we know that if a choice process is compressed and has an ABS representation $\tilde{C}_A(\mathcal{Z} \backslash \mathcal{Z}(M)) = 0 \ \forall \ A \in \mathcal{X}$, and so $\bar{C}_A\left(\mathcal{Z}(M)\right) = 1$. This in turn proves that $(X, \tilde{C})$ is finite.

To prove that a compressed SCP that satisfies for which$\succ^{\tilde{C}}$ and $\sim^{\tilde{C}}$ satisfy OWC is finite, note that this implies that the associated choice process must apply full measure to $\mathcal{Z}^{NCY}$, those elements of $\mathcal{Z}$ in which there are no cycles (the proof that $\mathcal{Z}^{NCY}$ is measurable is in appendix 2),

$$\mathcal{Z}^{NCY} = \{Z \in \mathcal{Z} | Z_{t+1} \neq Z_t \implies Z_{t+s} \neq Z_t \text{ any } s \geq 1\} \in \mathcal{F}.$$

To see why $\succsim_{\tilde{C}}$ satisfying OWC implies that $\tilde{C}_A(\mathcal{Z}^{NCY}) = 1$ for any set $A \in X$, assume to the contrary that there is a set of strictly positive measure according to some $A \in X$ such that $Z_{t+1} \neq Z_t$, yet $Z_{t+s} = Z_t$ for some $s \geq 1$. There are two possibilities. One is that there is an element $y \in Z_{t+1}$ with $y \notin Z_t$: in this case consider any $x \in Z_{t+1}$, and note that $\tilde{C}_A(R^{xy}) > 0$ due to exit of element y and entry of element x from period $t+1$ to period $t+s$, while also one of the statements $\tilde{C}_A(R^{yx}) > 0$ or $\tilde{C}_A(J^{yx}) > 0$ in consideration of the entry of y in period $t+1$. In the former case, the contradiction to $\succsim_{\tilde{C}}$ satisfying OWC is that $x \succ^{\tilde{C}} y$ and $y \succ^{\tilde{C}} x$, while in the latter case the contradiction is that $x \succ^{\tilde{C}} y$ and $y \sim^{\tilde{C}} x$. Alternatively, it could be that there is some $y \in Z_t$ and $y \notin Z_{t+1}$. A similar argument shows that this violates $\succsim^{\tilde{C}}$ satisfying OWC . This establishes the required finiteness, since elements of $\mathcal{Z}^{COM} \cap \mathcal{Z}^{NCY}$ are unchanging after a number of periods no larger than the cardinality of $\mathcal{X}$, completing the proof. ■

10.3 Structure of The Finite Case

Proposition 4 *A stochastic choice process $(X, \tilde{C})$ is finite if and only if it is the convex combination of a finite number of deterministic choice processes, in that there exist some J deterministic choice processes $\{(X, C^j)\}_{j=1}^{J}$ and weight vector $\lambda \in \mathbb{R}^J_{++}$ satisfying $\sum_{j=1}^{J} \lambda_j = 1$, and such that $\tilde{C} = \sum_{j=1}^{J} \lambda_j C^j$: i.e for all $F \in \mathcal{F}$ and $A \in \mathcal{X}$,*

$$\tilde{C}_A(F) = \sum_{j=1}^{J} \lambda_j C^j_A(F) = \sum_{j=1}^{J} \lambda_j 1_{\{C^j_A \in F\}} .$$

Proof. It is immediate that the convex combination of deterministic choice processes $\{(X, C^j)\}_{j=1}^{J}$ is finite, since $\tilde{C}_A\{Z \in \mathcal{Z} |\ \exists\ j \in \{1, .., J\}$ s.t. $Z = C^j\} = 1$ all $A \in \mathcal{X}$. To prove that any finite process $(X, \tilde{C})$ can be decomposed as the proposition asserts, use integers $1 \leq k \leq K$ to index elements Z_k of the finite set G with the property that $\tilde{C}_A(G) = 1\ \forall\ A \in \mathcal{X}$: we call these the basic choice processes. Since $\tilde{C}_A(Z_k) \geq 0$ and $\sum_{k=1}^{K} \tilde{C}_A(Z_k) = 1$ we can use indicator functions to record the probability of any set $F \in \mathcal{F}$ as a convex combination of these basic processes as follows,

$$\tilde{C}_A(F) = \sum_{k=1}^{K} \tilde{C}_A(Z_k) 1_{\{Z_k \in F\}}.$$

We now show that we can use these weights to construct a finite set of choice processes that are able simultaneously to capture such probability information across sets $F \in \mathcal{F}$ and $A \in \mathcal{X}$.

First, gather together in the finite set $\mathcal{J}$ all values taken on by the cumulative distributions taken in order according to k across all $A \in \mathcal{X}$,

$$\mathcal{J} = \left\{ x \in (0,1] | x = \sum_{i=1}^{k} \tilde{C}_A(Z_i) \text{ for some } A \in \mathcal{X}, k \in \{1, ..K\} \right\}.$$

We index members of the set $\mathcal{J}$ by $1 \leq j \leq J$ in increasing order, so that $x_j < x_{j+1}$, with $x_J = 1$. We now define a family of functions $f^A : \mathcal{J} \to G$ that, for each $A \in \mathcal{X}$, record which basic choice process is related to each cumulative probability level,

$$f^A(x_j) = \tilde{C}_A(Z_k) \text{ if and only if } x_j \in \left(\sum_{i=1}^{k-1} \tilde{C}_A(Z_i), \sum_{i=1}^{k} \tilde{C}_A(Z_i) \right].$$

We use these objects to construct the finite set of choice processes of interest using the following iteration. The probability assigned to the first deterministic choice process C^1 is x_1 and the actual specification involves using the set specific weights as follow,

$$C_A^1 = f^A(x_1).$$

If $J > 1$, we iterate the construction, using at step j weight $x_j - x_{j-1} > 0$ and specifying choice process C_A^j to satisfy,

$$C_A^j = f^A(\sum_{i=1}^{j} x_i).$$

The above construction identifies a finite set of deterministic choice process C^j, $1 \leq j \leq J$ and weights $\lambda_j = x_j - x_{j-1} \geq 0$ and summing to 1. We now such that, for all $A \in \mathcal{X}$ and $F \in \mathcal{F}$,

$$\tilde{C}_A(F) = \sum_{j=1}^{J} \lambda_j C_A^j(F) = \sum_{j=1}^{J} \lambda_j 1_{\{C_A^j \in F\}}.$$

We consider first the sets $Z_k \in \mathcal{F}$, noting that,

$$\sum_{j=1}^{J} \lambda_j 1_{\{C_A^j = Z_k\}} = \sum_{j=1}^{J} \lambda_j 1_{\{f^A(\sum_{i=1}^{j} \lambda_i) = Z_k\}},$$

and that $f^A\left(\sum_{i=1}^{j}\lambda_i\right) = Z_k$ if and only if $\sum_{i=1}^{j}\lambda_i \in \left(\sum_{i=1}^{k-1}\tilde{C}_A(Z_i), \sum_{i=1}^{k}\tilde{C}_A(Z_i)\right]$. Hence we can identify j, l such $\sum_{i=1}^{j}\lambda_i = \sum_{i=1}^{k-1}\tilde{C}_A(Z_i)$ and $\sum_{i=1}^{l}\lambda_i = \sum_{i=1}^{k}\tilde{C}_A(Z_i)$, so that by construction we get,

$$\sum_{j=1}^{J}\lambda_j 1_{\left\{C_A^j = Z_k\right\}} = \sum_{i=1}^{k}\tilde{C}_A(Z_i) - \sum_{i=1}^{k-1}\tilde{C}_A(Z_i) = \tilde{C}_A(Z_k).$$

That the same is true for any $F \in \mathcal{F}$ follows directly, since,

$$\tilde{C}_A(F) = \sum_{i=1}^{K}\tilde{C}_A(Z_k)1_{\{Z_k \in F\}} = \sum_{i=1}^{K}\left(\sum_{j=1}^{J}\lambda_j 1_{\left\{C_A^j = Z_k\right\}}\right)1_{\{Z_k \in F\}} = \sum_{j=1}^{J}\lambda_j 1_{\left\{C_A^j \in F\right\}}.$$

■

10.4 Proof of Theorem 3

Proof. Application of the compression and decompression relations establishes that the finite case is all that needs to be considered. To prove that if $\sim^{\tilde{C}}$ and $\succ^{\tilde{C}}$ satisfy OWC ABS follows, we apply lemma 1 directly to show that $\succsim^{\tilde{C}}$ satisfying OWC implies existence of $\tilde{u} : X \to \mathbb{R}$ that respects the binary relations $\sim^{\tilde{C}}$ and $\succ^{\tilde{C}}$. Moreover, in light of the last proposition, $(X, \tilde{C})$ is the weighted average of deterministic choice processes, $\tilde{C} = \sum_{j=1}^{J}\lambda_j C^j$, which have the property that their corresponding relations $\sim^j$ and $\succ^j$ are all respected by the same $\tilde{u} : X \longrightarrow \mathbb{R}$, since $\sim^{\tilde{C}}$ and $\succ^{\tilde{C}}$ represent the union of these deterministic relations:

$$\begin{aligned}\tilde{C}_A(J^{xy}) &> 0 \text{ if and only if } x \sim^j y, \text{ some } 1 \le j \le J;\\ \tilde{C}_A(F^{xy}) &> 0 \text{ if and only if } x \succ^j y, \text{ some } 1 \le j \le J.\end{aligned}$$

Re-application of lemma 1 to each of the deterministic choice processes $\left\{(X, C^j)\right\}_{j=1}^{J}$ implies that $\sim^j$ and $\succ^j$ satisfy OWC for all j, and moreover that the utility function $\tilde{u} : X \to \mathbb{R}$ forms part of some ABS representation of them, further ensuring the existence of deterministic search processes S^j such that $(\tilde{u}, S^j)$ form ABS representations of (X, C^j) for all $1 \le j \le J$. Defining the corresponding weighted average search process $\tilde{S} \equiv \sum_{j=1}^{J}\lambda_j S^j$ and $v_A^{S^j} = \left\{\arg\max_{x \in S_A^j(t)} u(x)\right\}_{t=1}^{\infty}$, one can immediately confirm that $(\tilde{u}, \tilde{S})$ form a stochastic ABS representation of $(X, \tilde{C})$, since given $F \in \mathcal{F}$ and $A \in \mathcal{X}$,

$$\tilde{C}_A(F) = \sum_{j=1}^{J}\lambda_j 1_{\left\{C_A^j \in F\right\}} = \sum_{j=1}^{J}\lambda_j 1_{\left\{v_A^{S^j} \in F\right\}}.$$

But as $\tilde{S}_A \left\{ Z \in \mathcal{Z} | Z = S_A^j \text{ for no } j \in \{1,..,J\} \right\}$, we know that,

$$\begin{aligned} & \tilde{S}_A \left(\left\{ Z \in \mathcal{Z} | \left\{ \arg \max_{x \in Z_t} u(x) \right\}_{t=1}^{\infty} \in F \right\} \right) \\ = & \sum_{j=1}^{j} \tilde{S}_A(\tilde{S}_A^j) 1_{\left\{ v_A^{Sj} \in F \right\}} \\ = & \sum_{j=1}^{J} \lambda_j 1_{\left\{ v_A^{Sj} \in F \right\}}. \end{aligned}$$

The last equality follows from the fact that, $\forall\ j \in \{1,..J\}$, $\tilde{S}_A(\tilde{S}_A^J) = \lambda_j$.

To prove that ABS implies that $\sim^{\tilde{C}}$ and $\succ^{\tilde{C}}$ satisfy OWC, note that if $(\tilde{u}, \tilde{S})$ form an ABS representation of $(X, \tilde{C})$, Lemma 1 then implies that $\tilde{u}$ respects the orderings $\sim^{\tilde{C}}$ and $\succ^{\tilde{C}}$ on X, which therefore satisfy OWC. ■

10.5 Proof of Theorem 4

As for ABS, the proof need be given only for the finite case in light of the compression and decompression operations. This finite proof follows from the a generalized version of the RBS characterization precisely as the deterministic result followed from proposition 1. To prove the relevant result we need to generalize the ordering $\succsim^{\bar{R}}$ of section 4.

Definition 18 *Given a stochastic choice process $(\tilde{X}, C)$ and set $D \in \mathcal{X}$, the binary relation $\succ_D^{\tilde{L}}$ on X is defined by $x \succ_D^{\tilde{L}} y$ if $\{x \cup y\} \cap D \neq \emptyset$, and there exists $A \in \mathcal{X}$ with $x, y \in A$ with $\tilde{C}_A^L \{x\} > 0$ and $\tilde{C}_A^L \{y\} = 0$. The binary relation $\succ^{\tilde{R}}$ is defined as $\succ_D^{\tilde{L}} \cup \succ^{\tilde{C}}$.*

Proposition 5 *A finite stochastic choice process model $(X, \tilde{C})$ has a stochastic RBS representation $(u, \tilde{S}, \rho)$ with below-reservation set $D \subset X$ if and only if :*

1. *$\tilde{X}^N \subset D$.*
2. *If $x \in D$ and $x \succsim_D^{\bar{R}} y$, then $y \in D$.*
3. *Given $x_1, x_2, x_3, .., x_n \in X$ with $x = x_1 \succsim_D^{\bar{R}} x_2 \succsim_D^{\bar{R}} .. \succsim_D^{\bar{R}} x_n = x$, there is no k with $x_k \succ_D^{\bar{R}} x_{k+1}$.*

Proof. The proof that conditions (1) - (3) of the proposition are sufficient is constructive, and similar to that in the deterministic case. As there, we define a utility function $u : X \to R$ that respects $\succ_D^{\tilde{R}}$ and $\sim$ on X, define reservation utility ρ as the average between the maximum on the set D and the minimum on the set $X \backslash D$, and demonstrate again that $X \backslash D$ is the reservation set associated with the utility function $u : X \to R$ and reservation utility level ρ by noting that $u(x) > u(y)$ whenever $x \in X \backslash D$ and $y \in D$. To see this, note that $x \in X \backslash D$ and $y \in D$ implies by condition (2) above that $C^L_{\{x,y\}}(\{x\}) = 1$, whereupon $x \succ_D^{\tilde{R}} y$, so that $u(x) > u(y)$ by construction.

We now consider all deterministic processes C^j in the decomposition of the finite stochastic choice process map $\tilde{C}$ that we know by the last proposition to be available. Define X_j^N as the non-terminal set associated with deterministic choice process (X, C^j), and define also the corresponding binary relations $\sim^j$, $\succ^{C^j}$, $\succ_D^{L^j}$, $\succ_D^{R^j}$, $\succsim^{C^j}$, $\succsim_D^{L^j}$, and $\succsim_D^{R^j}$. We show now that any set $D \subset X$ with properties 1-3 above for the stochastic choice process $(X, \tilde{C})$ necessarily satisfies corresponding deterministic properties 1-3 established in theorem 2 to be necessary and sufficient for D to be a reservation set in some RBS representation of each (X, C^j). With respect to the first such property, note directly from the definition that any non-terminal element in (X, C^j) is necessarily so in the stochastic models, so that $X_j^N \subset \tilde{X}^N$, hence $X_j^N \subset D$ as required. The second and third properties follow directly from the fact that, for any $j \in \{1, ..., J\}$, $x \succ_D^{R^j} y \Rightarrow x \succ_D^{\bar{R}} y$ and $x \sim^j y \Rightarrow x \sim y$. To see this, note first that $x \succ_D^{R^j} y$ implies that either $x \succ^{C^j} y$ or $x \succ_D^{L^j} y$. The former case indicates that for some $A \in \mathcal{X}$, $\tilde{C}_A(R^{xy}) \geq \lambda_j > 0$, and so $x \succ^{C^j} y$, while the latter implies that, for some $A \in \mathcal{X}$ and $B \subset A$, $x \in B$, $y \notin B$ and $\tilde{C}_A^L(B) \geq \lambda_j > 0$, so $x \succ_D^L y$. In each case, $x \succ_D^{\bar{R}} y$. A similar argument shows that $x \sim^j y$ implies for some $A \in \mathcal{X}$, $\tilde{C}_A(J^{xy}) \geq \lambda_j > 0$ and so $x \sim y$. This result shows that any violation of conditions 2 and 3 at the level of the deterministic choice process j would lead to a violation of the equivalent condition at the level of the stochastic choice function.

Given that the assumptions of theorem 2 are satisfied, we conclude not only that there exists an RBS representation of each (X, C^j) with reservation set D, but also that the utility function $u : X \to R$ and reservation utility level ρ can be utilized in constructing such a representation, given that these are precisely the objects that are constructed in the course of the deterministic proof. Hence, for each j, there exists a search correspondence S^j such that (u, S^j, ρ) represents an RBS representation of (X, C^j). We show now that $(u, \tilde{S}, \rho)$ comprises an RBS representation of

$(X, \tilde{C})$, where $\tilde{S}$ is the corresponding convex combination of the deterministic search processes S^j,

$$\tilde{S} = \sum_{j=1}^{J} \lambda_j S^j$$

That $(u, \tilde{S})$ for a stochastic ABS representation follows as in the proof of the ABS representation theorem. That $X \backslash D = \{x \in X | u(x) \geq \rho\}$ holds by construction. Moreover given $A \in \mathcal{X}$, we know that if $A \cap (X \backslash D) = \phi$, then A is searched fully in all search correspondences S^j, ensuring that $\tilde{S}_A^L(A) = 1$. On the other hand, if $A \cap X^R \cap (X \backslash D) \neq \phi$, then we know that in the limit, search reaches into the reservation set in all search correspondences S^j, ensuring that $\tilde{S}_A \left\{ Z \in \mathcal{Z} | H^R(Z) \text{ is finite} \right\} = 1$. Finally, since each element in the reservation set has the property that search ceases at once with probability one when such an element is encountered in each S^j, we know that $\tilde{S}_A \left\{ Z \in \mathcal{Z} | \tilde{S}_A^L = \tilde{S}_A \left(H^R(Z) \right) \right\} = 1$, completing the proof that $(u, \tilde{S}, \rho)$ comprises an RBS representation of $(X, \tilde{C})$.

The proof that conditions 1-3 above are necessary for a finite stochastic choice process $(X, \bar{C})$ to have an RBS representation $(u, \tilde{S}, \rho)$ is essentially identical to that in the deterministic case. We let D be the below reservation set generated by that representation, and establish that the three conditions of the proposition hold. ■

Proof of Theorem 5 Application of Lemma 1 translates the theorem to the statement that there exists $u : X \to \mathbb{R}$, $\rho : \Gamma \to \mathbb{R}$, and $\Theta : \Gamma \to \bar{\mathcal{S}}$ such that $(u, \Theta(\gamma), \rho(\gamma))$ forms a stochastic RBS representation of $\Phi(\gamma)\ \forall\ \gamma \in \Gamma$ iff there exists $v : X \to \mathbb{R}$ that respects $\succ^{\tilde{R}(\Gamma)}$ and $\sim^{\tilde{C}(\Gamma)}$. To see that existence of such a function $v : X \to \mathbb{R}$ is necessary, note from theorem 4 that the given function $u : X \to \mathbb{R}$ such that $(u, \Theta(\gamma), \rho(\gamma))$ forms a stochastic RBS representation of $\Phi(\gamma)$ for all $\gamma \in \Gamma$ respects $\succ^{\tilde{R}(\gamma)}$ and $\sim^{\tilde{C}(\gamma)}$ all $\gamma \in \Gamma$ and hence respects $\succ^{\tilde{R}(\Gamma)}$ and $\sim^{\tilde{C}(\Gamma)}$. Conversely, given $v : X \to \mathbb{R}$ that respects $\succ^{\tilde{R}(\Gamma)}$ and $\sim^{\tilde{C}(\Gamma)}$, by definition it respects $\succ^{\tilde{R}(\gamma)}$ and $\sim^{\tilde{C}(\gamma)}$ all $\gamma \in \Gamma$, whereupon theorem 4 implies that there exists an RBS representation of $\Phi(\gamma)$ for all $\gamma \in \Gamma$. In fact the proof of theorem 4 reveals that the given function $v : X \to \mathbb{R}$ that respects $\succ^{\tilde{R}(\gamma)}$ and $\sim^{\tilde{C}(\gamma)}$ can form the basis for an ABS representation with appropriately defined $\rho : \Gamma \to \mathbb{R}$ and $\Theta : \Gamma \to \bar{\mathcal{S}}$, with $(v, \Theta(\gamma), \rho(\gamma))$ therefore forming the required stochastic RBS representation of $\Phi(\gamma)\ \forall\ \gamma \in \Gamma$.

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