Strongly nonmanipulable multi-valued collective choice rules*

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Introduction

A collective choice rule is 'manipulated' whenever some individual misrepresents his preferences in order to secure an outcome preferred to the outcome when he his honest. According to the theorem of Gibbard [6] and Satterthwaite [10] (G-S in what follows), the only collective choice rule that cannot be manipulated is a dictatorship. While this result looks overwhelmingly final, and depressing, it does raise a few questions. In G-S, a collective choice rule is single-valued: for each specification of all individuals' preferences, or each 'preference profile', it produces a single winner, a unique alternative. What if, as it is often the case, the collective choice rule allows for ties? What happens to the nonmanipulability implies dictatorship theorem then?

These are essentially the questions asked by Kelly [9] and Barbera [2]. If a collective choice rule is multi-valued, however, a new issue arises. Properly defined, alternatives are mutually exclusive, and ultimately only one can be chosen. A multi-valued collective choice rule is then a first step; if it produces ties, the ties must be broken. The obvious tie-breaker is a coin toss, that is, a random device. What happens to the nonmanipulability implies dictatorship theorem when the collective choice rule is random, or when it is a mixture of nonrandom and random parts? This is the question asked by Gibbard in [7].

The answers of Kelly [9] and Barbera [2] can be loosely characterized this way: Nonmanipilable multi-valued collective choice rules necessarily make some people 'weak dictators'. Now, a person is a 'weak dictator' if, whenever he prefers x to y *and* the social choice agenda is restricted to x versus y, x must at least tie y. However, it is arguable that the existence of weak dictatorship is per se objectionable; since the welfare economist's mainstay, the Pareto rule, makes everyone a weak dictator. In another

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paper, Barbera [1] does have a nonmanipulability implies *dictatorship* theorem for multi-valued collective choice rules. But in order to get that result, he is forced to assume two strong properties for the rule, unanimity and positive responsiveness, in addition to nonmanipulability.

Gibbard's [7] answer for random (or mixed) collective choice rules is more complicated. He shows that any nonmanipulable random rule must be of the following two-stage type: In stage 1, it randomly chooses either an individual i, or a pair of alternatives $\{x, y\}$. This choice is made according to fixed probabilities that do not depend on individual preferences over the alternatives. Then, if person i has been chosen in stage 1, in stage 2 a final winning alternative is chosen according to a random scheme that depends only on i's preferences. If the pair $\{x, y\}$ has been chosen in stage 1, in stage 2 a final winner is chosen from $\{x, y\}$ according to a random scheme that depends on all persons' preferences. Thus any random nonmanipulable collective choice rule is a 'probability mixture' of 'unilateral' schemes (i.e., schemes attached to single individuals) and 'duple' schemes (i.e., schemes in which a choice is made from a fixed pair of alternatives).

The purpose of this short paper is to prove that *any multi-valued collective choice rate that is nonmanipulable in a strong sense and non-imposed must be (1) a dictatorship or (2) a duumvirate.* That is, there must be a single individual who dictates the social choice, or there must be a pair of equally powerful individuals who dictate the social choice. The theorem is like Kelly's [9] and Barbera's [2], except it uses a substantially stronger definition of nonmanipulability and its conclusion is consequently stronger. The theorem in this paper also differs from Barbera's [1] result, because the strong conclusion of dictatorship there depends on powerful ancillary assumptions, while the strong conclusion of dictatorial or duumviral rule here depends on a strong notion of nonmanipulability. Moreover, the proof below is an easy consequence of Gibbard's theorem [7], and this paper, therefore, connects multi-valued choice rule analysis with explicit random (or mixed) choice rule analysis. I might add that the theorem could also be derived from Gibbard's Corollary I in [8].

A few words about assumptions are important here, in particular, about the nonmanipulability assumption. Manipulation is unambiguous for a single-valued rule: in that context, a person manipulates if, by falsely revealing his preferences, he brings about a single winner which he prefers to the single winner when he is honest. But when the outcome is a multi-valued choice set, or when the mechanism is random, the meaning of manipulation can be ambiguous. Several alternative definitions are given in [3] and [5]. In this paper I use a strong, but intuitively clear, definition of nonmanipulability: A collective choice rule maps preferences into choice sets. I assume that a person prefers choice set A to choice set B if an even-chance lottery over A gives him a higher expected utility than an even-chance lottery over B (see, e.g., Fishburn [4]). If a person prefers the choice set produced when he is dishonest about his preferences to the one produced when he is honest, I say he 'manipulates' the collective choice rule. The collective choice rule is 'nonmanipulable' if, no matter what people's preferences are, and no matter what utility functions represent those preferences, it cannot be manipulated by anyone. That is, for a nonmanipulable rule, no one can ever increase his expected utility from an even-chance lottery over the choice set by lying about his preferences. This idea of nonmanipulability is identical to Gibbard's [7] idea of strategy-proofness, except the latter is not restricted to the even-chance lottery context. It is also, I might add, identical to my 'definition 7 nonmanipulability' in [3].

In addition to nonmanipulability, I assume that the collective choice rule meets a mild-looking non-imposition assumption: For any profile x, there is some circumstance, some preference profile, under which $\{x\}$ is the choice set. And, like Gibbard [7], I assume no indifference is allowed. As Gibbard argues, if there is a dictatorship for preference profiles with no indifference, then the collective choice rule is flawed – even though that dictatorship is modified for preference profiles with indifference.

The model and results

There is a finite set X of alternatives, and n people, indexed by $i = 1, 2, \ldots, n$. R_i is i's preference order. I assume that no indifference is allowed; all preference orders are antisymmetric, as well as complete, reflexive, and transitive. $R = (R_1, R_2, \ldots, R_n)$ is a *preference profile*. $C(\cdot)$ is a multi-valued *collective choice rule*; it maps preference profiles into choice sets. For the preference profile R, $C(R)$ is the corresponding choice set.

The utility function $u_i(\cdot)$ is said to *represent* R_i if $u_i(x) \geqslant u_i(y)$ iff xR_iy, for all $x, y \in X$. $C(\cdot)$ is *manipulated by i at R* if there exists a utility function $u_i(\cdot)$ that represents R_i, and a 'false' preference relation R_i' such that

$$\frac{1}{|C(R_i)|} \sum_{x \in C(R_i)} u_i(x) < \frac{1}{|C(R_i')|} \sum_{x \in C(R_i')} u_i(x) \ .$$

Here $C(R_i) \equiv C(R_1, R_2, \ldots, R_i, \ldots, R_n)$, and $C(R_i') \equiv C(R_1, R_2, \ldots, R_i', \ldots, R_n)$. $C(\cdot)$ is *nonmanipulable* if it can never be manipulated by any i.

$C(\cdot)$ is *non-imposed* if for every $x \in X$, there exists a preference profile R such that $C(R) = \{x\}$.

Let $M_i(R_i)$ be i's most preferred alternative: $M_i(R_i) \equiv \{x \in X \mid xR_iy$ for all $y \in X, y \neq x\}$. $C(\cdot)$ is a *dictatorship* if there exists an i such that $C(R) = M_i(R_i)$ for all R. $C(\cdot)$ is a *duumvirate* if there is a pair of individuals i and j such that $C(R) = M_i(R_i) \cup M_j(R_j)$ for all R.

Before proceeding to the theorem, it is useful to establish a simple preliminary result.

Lemma: Let R be a preference profile in which everyone ranks alternative x first. If $C(\cdot)$ is non-imposed and nonmanipulable, then $C(R) = \{x\}$.

Proof: Suppose to the contrary that $C(R) \neq \{x\}$. Since $C(\cdot)$ is non-imposed, there exists a profile R^1 for which $C(R^1) = \{x\}$. Consider the following choice sets:

$$C(R_1, R_2, \ldots, R_n) \quad (\neq \{x\})$$
$$C(R_1^1, R_2, \ldots, R_n)$$
$$C(R_1^1, R_2^1, \ldots, R_n)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$C(R_1^1, R_2^1, \ldots, R_n^1) \quad (= \{x\})$$

Clearly, for some i between 1 and n,

$$C(R_1^1, \ldots, R_{i-1}^1, R_i, \ldots, R_n) \neq \{x\},$$

while

$$C(R_1^1, \ldots, R_{i-1}^1, R_i^1, \ldots, R_n) = \{x\}.$$

This obviously permits i to manipulate $C(\cdot)$, a contradiction. Q.E.D.

Now I turn to the theorem, which, as I have already noted, is a simple consequence of Gibbard's [7] theorem. I will assume an arbitrary collective choice rule $C(\cdot)$ is nonmanipulable and non-imposed. I will also assume there are at least three alternatives. (If there are only two alternatives in X, majority rule is a flawless multi-valued collective choice rule.) The upshot of these assumptions is that, *at best,* $C(\cdot)$ places all power in the hands of only two people.

Theorem: Suppose $C(\cdot)$ is nonmanipulable and non-imposed. Assume $|X| \geq 3$. Then $C(\cdot)$ is either a dictatorship or a duumvirate.

Proof: First a word about strategy. I start with a given multi-valued $C(\cdot)$ that is nonmanipulable and non-imposed. With my given $C(\cdot)$, I define a Gibbard decision scheme d. Because $C(\cdot)$ is nonmanipulable in the sense of this paper, it follows that d must be strategy-proof in Gibbard's [7] sense. Consequently, I can use Gibbard's theorem to conclude that d is a probability mixture of unilateral and duple decision schemes. Next I use the mild-looking but powerful non-imposition assumption on $C(\cdot)$, to show d must actually be a probability mixture of unilateral decision schemes. Finally I use the Lemma to show that d must be a probability mixture of at most two unilateral decision schemes, and it follows that my original rule $C(\cdot)$ is either dictatorial or duumviral.

Now to the proof: $C(\cdot)$ maps preference profiles into multi-valued choice sets. A *decision scheme* maps preference profiles into probability distributions over the set of alternatives. Use $C(\cdot)$ to define an even-chance decision scheme:

$$d(x, R) \equiv \begin{cases} \dfrac{1}{|C(R)|}, & \text{for } x \in C(R) \\ 0, & \text{for } x \neq C(R) \end{cases}$$

Since $C(\cdot)$ is nonmanipulable, d must be strategy-proof in Gibbard's [7] sense.

Now a decision scheme is *unilateral* if it depends only on one person's preference relation; let $d_i(\cdot, R_i)$ denote the unilateral scheme associated with person i. A decision scheme is *duple* if, for all preference profiles, it assigns probabilities of zero to all alternatives outside a given pair; let $d_{xy}(\cdot, R)$ denote the duple scheme associated with the pair $\{x, y\}$. Then $d_{xy}(z, R) = 0$ for $z \notin \{x, y\}$. The pair $\{x, y\}$ is not ordered; so there is no d_{yx} distinct from d_{xy}; also, it is possible to have $x = y$.

By the 'weak version' of Gibbard's theorem [7], d must be a *probability mixture* of unilateral and duple decision schemes. This means that there are nonnegative fixed weights $\{\alpha_i\}$, where $i = 1, \ldots, n$, and $\{\alpha_{xy}\}$, where $\{x, y\} \subset X$, such that (1) the weights sum to 1, and (2) for all x and R,

$$d(x, R) = \sum_{i=1}^{n} \alpha_i d_i(x, R_i) + \sum_{y \in X} \alpha_{xy} d_{xy}(x, R) .$$

Now I claim that for any given pair $\{x, y\}$, $\alpha_{xy} = 0$. By assumption, there are at least three alternatives. Take $z \notin \{x, y\}$. By the non-imposition assumption there is a preference profile R such that $C(R) = \{z\}$. By the definition of d, $d(x, R) = d(y, R) = 0$. Since $d_{xy}(x, R) + d_{xy}(y, R) = 1$, and since $d(x, R)$ and $d(y, R)$ are sums of non-negative numbers plus $\alpha_{xy} d_{xy}(x, R)$ and $\alpha_{xy} d_{xy}(y, R)$, respectively, α_{xy} must be zero.

Consequently, d is in fact a probability mixture of unilateral decision schemes: For all x and R,

$$d(x, R) = \sum_{i=1}^{n} \alpha_i d_i(x, R_i), \quad \text{where } \sum_{i=1}^{n} \alpha_i = 1 .$$

Next, I claim that, if $\alpha_i > 0$, the unilateral scheme d_i must always attach all probability to i's first choice. Let R be a preference profile in which everyone ranks x first. By the Lemma and the definition of d,

$$d(x, R) = \sum_{i=1}^{n} \alpha_i d_i(x, R_i) = 1$$

Since

$$\sum_{i=1}^{n} \alpha_i = 1$$

and

$$d_i(x, R_i) \leqslant 1 \text{ for all } i, d_i(x, R_i) = 1 \text{ for all } i \text{ for whom } \alpha_i > 0 .$$

Next I claim that if $\alpha_i > 0$ for more than one i, then $\alpha_i > 0 \Rightarrow \alpha_i = 1/2$. Suppose without loss of generality that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$, while $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n = 0$. Consider a preference profile R in which person 1 ranks x first, persons 2 through k rank $y \neq x$ first, and the rankings of $k + 1$ through n are arbitrary. Then

$$d(x, R) = \alpha_1 \cdot 1 > 0 \ ,$$

and

$$d(y, R) = \sum_{i=2}^{k} \alpha_i \cdot 1 > 0 \ ,$$

while

$$d(z, R) = 0 \text{ for all } z \notin \{x, y\}.$$

But by the construction of d, $d(x, R) > 0$, $d(y, R) > 0$, and $d(z, R) = 0$ for all $z \notin \{x, y\}$, implies $d(x, R) = d(y, R) = ½$. Therefore, $\alpha_1 = ½$. But this argument clearly applies to any individual for whom $\alpha_i > 0$. Since $\Sigma \alpha_i = 1$, there can be only *two* such i's, and these two constitute a duumvirate.

On the other hand, if $\alpha_i > 0$ for only one i, then $\alpha_i = 1$ and that i is a dictator. Q.E.D.

An example illustrates what the theorem says about dictatorships, duumvirates, and, e.g., *triumvirates*, or triple-dictatorships. Suppose there are three individuals and three alternatives $\{x, y, z\}$. If person 1 is a dictator, then the choice set must always be 1's best alternative. If 1 and 2 are a duumvirate, then the choice set must always be the union of 1's best alternative and 2's best alternative. If 1, 2, and 3 are a triumvirate, then the choice set must always be the union of the three individuals' best alternatives.

Let the preference profile R be given by:

1. $x \quad y \quad z$
2. $z \quad y \quad x$

3. $x \quad y \quad z$

(That is, person 1 prefers x to y to z, and so on.) Now if $C(\cdot)$ is a *triumvirate*, $C(R) = \{x, z\}$. But person 3 could then manipulate by falsely declaring his preferences to be R_3': $y \; x \; z$. If he does this, $C(R_3') = \{x, y, z\}$, which gives 3 a higher expected utility than $C(R_3) = C(R) = \{x, z\}$, whenever

$$u_3(y) > \frac{u_3(x) + u_3(z)}{2} \quad .$$

On the other hand, if $C(\cdot)$ is a *duumvirate*, such manipulation is impossible. For if person 3 is not a member of the duumvirate, he can't affect the choice set. If he is a member of the duumvirate, then his best alternative is in the choice set, and there can be at most one other alternative in that set, which he cannot affect. So he can't insert another alternative in the choice set by lying, and manipulation along the above lines is again impossible.

Finally, if $C(\cdot)$ is a *dictatorship*, it obviously can't be manipulated by the dictator or by his subjects.

The theorem establishes what this example suggests: If $C(\cdot)$ is nonimposed and nonmanipulable, it must be a dictatorship or a duumvirate. It cannot give power to more than 2 people.

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