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An axiomatization of the inner core

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Abstract. We axiomatize the inner core in a similar way as the one proposed by Aumann (1985) in order to characterize the NTU value.

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1. Introduction

Shapley (1969) proposed a procedure that allows to extend TU-solution concepts to some larger class of NTU games[2]. The so-called fictitious-transfer procedure has mostly been applied to the Shapley TU-value (cf. the NTU value). Nevertheless, it has also been applied successfully to the TU-core, thus defining what is called the inner core. In particular, the inner core is essentially equivalent to the set of feasible allocations that are not dominated by some randomized blocking plan (cf. theorem 9.5 of Myerson (1991), Qin (1993) and de Clippel and Minelli (2002)), it is nonempty for games that satisfy a strong balancedness condition (cf. theorem 9.6 of Myerson (1991) and Qin (1994a)), and it is equivalent to the core and the set of competitive equilibria in well-behaved large economies (cf. Qin (1994b)). In addition, Professor Myerson has extended the analysis of the inner core to incomplete information frameworks (cf. section 10.9 of Myerson (1991) and Myerson (1995)).

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[2] TU (resp. NTU) stands for transferable (resp. non-transferable) utility.

In the present note, we axiomatize the inner core in a similar way as the one proposed by Aumann (1985) in order to characterize the NTU value. We compare in detail the two axiomatic systems. We observe that the conditional sure-thing property is a key axiom to be satisfied for this approach to work. As a consequence, we must restrict ourselves to games for which the feasible set of the grand coalition satisfies some smoothness property. On the other hand, we explain why the approach cannot be adapted for characterizing the extension of any TU-solution concept to some larger class of NTU-games via the Shapley procedure. The core satisfies all the axioms characterizing the inner core, except the conditional sure-thing property.

2. Definitions

Let n be a positive integer, let $N := \{1, \ldots, n\}$ be the set of players, and let $P(N)$ be the set of coalitions, that is the set of nonempty subsets of N. Elements of $\mathbb{R}^N$ that describe a distribution of payoffs for the players are called allocations. For each $S \in P(N)$, $\mathbb{R}^S$ is seen as a subset of $\mathbb{R}^N$: $\mathbb{R}^S := \{x \in \mathbb{R}^N \mid (\forall i \in N \backslash S) : x_i = 0\}$. $\mathbb{R}^N_+$ (resp. $\mathbb{R}^N_{++}$) denotes the set of vectors in $\mathbb{R}^N$ with nonnegative components (resp. positive components). For each pair of vectors (λ, x) in $\mathbb{R}^N$, λx denotes the vector $(\lambda_1 x_1, \ldots, \lambda_n x_n)$. This expression should be distinguished from the inner product $\lambda.x := \sum_{i \in N} \lambda_i x_i$. Moreover, $\frac{x}{\lambda}$ stands for $\left(\frac{x_1}{\lambda_1}, \ldots, \frac{x_n}{\lambda_n}\right)$ (whenever each component of λ is different from zero). If X is a subset of $\mathbb{R}^N$, then the frontier of X (denoted ∂X) is the intersection of the closure of X with the closure of its complement, that is $\partial X := \overline{X} \cap (\overline{\neg X})$. A set X is comprehensive if $x \in X$ whenever there exists some $x' \in X$ such that $x \leq x'$. A convex and comprehensive set X is *positively smooth* if it admits a unique supporting hyperplane with a strictly positive normalized orthogonal vector, at each point of its frontier.

A *game* V is a function that associates to each coalition S a nonempty proper subset $V(S)$ of $\mathbb{R}^S$, such that

1. $V(N)$ is comprehensive, convex and positively smooth;
2. $dom(v^{\cdot}(N)) \subseteq \bigcap_{S \in P(N)} dom(v^{\cdot}(S))$, where $dom(v^{\cdot}(S)) := \{\lambda \in \mathbb{R}^N_+ \backslash \{0\} \mid v^{\lambda}(S) := \sup_{x \in V(S)} \lambda.x < \infty\}$, for each $S \in P(N)$.

Condition 2 has been introduced by Kern (1985) under the name "normal closedness". It requires for the domain of definition of the support function of $V(N)$ to be included in the domain of definition of the support function of $V(S)$, for each $S \in P(N)$. The role of condition 2 is indicated in footnote 4 in the proof of our proposition. Our definition of game is essentially identical to the one proposed by Aumann (1985). Indeed, when $V(N)$ is comprehensive, the positive smoothness condition is equivalent to the conjunction of conditions (3.1) (smoothness) and (3.2) (non-levelness) of Aumann (1985). On the other hand, condition 2 is implied by condition (3.3) (weak monotonicity) of Aumann (1985).

Operations on games are defined from operations on sets, by applying them coalition by coalition. For example, $(V + W)(S) := V(S) + W(S)$ and $\overline{V}(S) := \overline{V(S)}$, for each $S \in P(N)$. Similarly, $V \subseteq W$ means that $V(S) \subseteq W(S)$, for each $S \in P(N)$.

A *TU-game* is a function $v : P(N) \to \mathbb{R}$. A game V is *equivalent to* a TU-game v if $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S)\}$, for each coalition S.

A *solution* is a function that associates to each game a set of allocations. For example, the *core* of a game V is the set of (almost) feasible allocations (i.e. in $\overline{V}(N)$) for which no coalition can strictly improve the utility of each of its members given its own opportunities of cooperation. Another example of solution is Shapley's *inner core*: an allocation x is in the inner core of a game V if and only if x is (almost) feasible and there exists $\lambda \in dom(v^{\cdot}(N))$ such that[3] $\lambda x \in c(v^{\lambda})$ (where c denotes the core defined for TU-games). The inner core of a game V will be denoted $IC(V)$. The inner core is included in the core.

3. Axiomatic characterization of the inner core

Here is a list of properties that could be imposed on a solution Σ.

Axiom 0 (Closure Invariance-CI) $\Sigma(V) = \Sigma(\overline{V})$, for each game V;
Axiom 1 (Efficiency-EFF) $\Sigma(V) \subseteq \partial V(N)$, for each game V;
Axiom 2 (Extension-EXT) $\Sigma(V) = c(v)$, for each game V and each TU-game v such that V is equivalent to v;
Axiom 3 (Scale Covariance-SC) $\Sigma(\lambda V) = \lambda \Sigma(V)$, for each $\lambda \in \mathbb{R}^N_{++}$ and each game V;
Axiom 4 (Conditional Decreasingness-CD) $\Sigma(W) \cap \overline{V}(N) \subseteq \Sigma(V)$, for each pair of games (V, W) such that $V \subseteq W$;
Axiom 5 (Conditional Sure-Thing-CST) $\Sigma(V) \cap \Sigma(W) \cap \partial(U(N)) \subseteq \Sigma(U)$, for each triple of games (V, W, U) such that $U = \frac{V+W}{2}$.

CI is a natural technical condition. EFF requires that any element of a solution should be Pareto optimal. EXT means that the solution has to coincide with the TU-core on the class of TU-games. This reproduces the point of view of Shapley (1969) (cf. also Myerson (1992)) who proposes to define solutions for general games by extending solutions defined for TU-games. SC imposes the solution to vary in covariance with positive linear transformations of the individual utilities. This is a necessary requirement if the allocations are the image of economic outcomes through utility functions (representing preferences) that are unique only up to positive linear transformations. CD requires that any allocation belonging to the solution of a game W which is (almost) feasible in a game V, should also belong to the solution of V, if the coalitions have no more opportunities of cooperation in V than in W. It sounds very natural if a solution is understood as determining a set of allocations that are not "blocked" in some reasonable sense. Finally, CST can be interpreted as follows: if an allocation x belongs to the solution of both the games V and W, and if the players play the game U which amounts to play V or W with an equal probability, then the allocation x should belong to the solution of U, except if the players can use the uncertainty to their mutual advantage. We refer to the next section for a further analysis of some axioms.

[3] We note that $v^{\lambda}(S) \in \mathbb{R}$, for each $S \in P(N)$, thanks to condition 2 in the definition of a game.

Proposition. *The inner core is the only solution that satisfies CI, EFF, EXT, SC, CD and CST.*

Proof: 1. We start by proving that the inner core satisfies the six axioms:

1.a) CI: Indeed, the support functions of $V(S)$ and of $\overline{V}(S)$ are the same, for each $S \in P(N)$.
1.b) EFF: Let $x \in IC(V)$. Then there exists $\lambda \in \mathbb{R}^N_{++}$ such that $x \in \arg\max_{y \in V(N)} \sum_{i \in N} \lambda_i y_i$. Hence, $x \in \partial V(N)$.
1.c) EXT: If V is equivalent to the TU-game v, then $dom(v^{\cdot}(N))$ is the set of utility weight vectors that are proportional to the unit vector $(1, \ldots, 1)$. Hence, $IC(V) = c(v)$.
1.d) SC: Let $W := \lambda V$. Let $x \in IC(W)$ and let $\mu \in \mathbb{R}^N_{++}$ be such that $\mu x \in c(w^\mu)$. It is easy to check that $w^\mu = v^{\mu\lambda}$. Then, $\mu x \in c(v^{\mu\lambda})$ or, equivalently, $\mu\lambda \frac{x}{\lambda} \in c(v^{\mu\lambda})$. So, $\frac{x}{\lambda} \in IC(V)$ and $x \in \lambda IC(V)$. As far as the other inclusion is concerned, let $x \in IC(V)$ and let $\mu \in \mathbb{R}^N_{++}$ be such that $\mu x \in c(v^\mu)$. It is easy to check that $v^\mu = w^{\mu/\lambda}$. Then, $\mu x \in c(w^{\mu/\lambda})$ or, equivalently, $\frac{\mu}{\lambda}\lambda x \in c(w^{\mu/\lambda})$. Hence, $\lambda x \in IC(W)$.
1.e) CD: Let $x \in IC(W) \cap \overline{V}(N)$ and let $\lambda \in \mathbb{R}^N_{++}$ be such that $\lambda x \in c(w^\lambda)$. As $V \subseteq W$, we have that $v^\lambda(S) \le w^\lambda(S)$ for each $S \in P(N)$. Hence, $x \in c(v^\lambda)$ and so $x \in IC(V)$.
1.f) CST: Let $x \in IC(V) \cap IC(W) \cap \partial(U(N))$ and let $\lambda \in \mathbb{R}^N_{++}$ be the unique normalized vector that is orthogonal to $U(N)$ at x. Then, x is optimal in $V(N)$ (resp. $W(N)$) and λ is orthogonal to $V(N)$ (resp. $W(N)$) at x. Hence, $\lambda x \in c(v^\lambda) \cap c(w^\lambda)$ (remember the smoothness assumption). This implies that $\lambda x \in c\left(\frac{v^\lambda + w^\lambda}{2}\right) = c(u^\lambda)$. So, $x \in IC(U)$.

2. Let now Σ be a solution that satisfies the six axioms.

2.a) We first show that $IC \subseteq \Sigma$. To this end, let V be a game and let $x \in IC(V)$. Then, there exists $\lambda \in \mathbb{R}^N_{++}$ such that $x \in \overline{V}(N) \cap \frac{c(v^\lambda)}{\lambda}$. By EXT, this latter expression amounts to $x \in \overline{V}(N) \cap \frac{\Sigma(V^\lambda)}{\lambda}$, where V^λ is the game equivalent to the TU-game v^λ. By CD, this implies that $x \in \frac{\Sigma(\lambda V)}{\lambda}$, since $\lambda V \subseteq V^\lambda$. Finally, by SC, we obtain that $x \in \Sigma(V)$.
2.b) We now show that $\Sigma \subseteq IC$. To this end, let V be a game and let $x \in \Sigma(V)$. By EFF, x is optimal in $V(N)$. Let $\lambda \in \mathbb{R}^N_{++}$ be the unique normalized vector that is orthogonal to $V(N)$ at x, let w be the TU-game defined by:

$$w(S) := \sum_{i \in S} \lambda_i x_i,$$

for each $S \in P(N)$, and let W be the game that is equivalent to w. By EXT, $\Sigma(W) = c(w) = \{\lambda x\}$. By SC, $\lambda x \in \Sigma(\lambda V)$. So, by CST, $\lambda x \in \Sigma\left(\frac{\lambda V + W}{2}\right)$. By CI, $\lambda x \in \Sigma\left(\overline{\frac{\lambda V + W}{2}}\right)$. On the other hand, $\overline{\frac{\lambda V + W}{2}}$ is equivalent to the TU-game u defined by[4]:

[4] We note that $u(S) \in \mathbb{R}$, for each $S \in P(N)$, since $\lambda \in dom(v^{\cdot}(N))$ and $dom(v^{\cdot}(N)) \subseteq \bigcap_{S \in P(N)} dom(v^{\cdot}(S))$ (cf. definition of a game).

$$u(S) := \frac{v^{\lambda}(S) + w(S)}{2},$$

for each $S \in P(N)$. So, by EXT, $\lambda x \in c(u)$. This implies that $\lambda x \in c(v^{\lambda})$.
QED.

4. Comments

1. The techniques used in the present note are very similar to the ones introduced by Aumann (1985) in order to characterize the NTU value. Let us compare the two axiomatic systems. CI, EFF, SC and CST are directly taken from Aumann's paper. On the other hand, EXT (resp. CD) can be understood as replacing Aumann's unanimity (resp. independence over irrelevant alternatives (IIA) and non-emptiness) axiom(s).

Aumann doesn't need to impose a kind of extension property (although he could have done it; cf. Myerson (1991)'s presentation of Aumann's result in section 9.9), but instead derives it from the other axioms. We didn't try to follow this path, since there doesn't exist an axiomatization of the TU-core that is as convincing as for the Shapley TU-value.

The non-emptiness axiom (which thus requires a further restriction of the set of admissible games) allows Aumann to impose only IIA, instead of CD[5]. Nevertheless, in our context, imposing non-emptiness would sound strange since the TU-core is empty for many TU-games. In addition, CD is a very natural axiom for solutions that determine a set of feasible allocations that are not "blocked".

2. Aumann's axiomatic system or the variation we propose cannot be used to axiomatize the extension of any TU-solution concept σ to the set of all games *via* the Shapley procedure. Indeed, the approach works only if σ satisfies itself the axioms on the class of TU-games. In particular, we must have:

1. $\sum_{i \in N} \sigma_i(v) = v(N)$, for each TU-game v;
2. $\sigma(\lambda v) = \lambda \sigma(v)$, for each $\lambda \in \mathbb{R}_{++}$ and each TU-game v;
3. $\sigma(v) \cap \sigma(w) \subseteq \sigma\left(\frac{v+w}{2}\right)$, for each pair (v, w) of TU-games.

The TU-core indeed satisfies these three properties. Nevertheless, property 3 can equivalently be replaced by an additivity condition[6] when we focus on TU-solution concepts that are single-valued and translation covariant[7]. As we know, this additivity property is very specific to the Shapley TU-value. In particular the nucleolus violates it. Indeed, Megiddo (1974) gave an example of a game v such that the payoff of some player as specified by the nucleolus for the game w, obtained from v by adding one to $v(N)$, is strictly less than the payoff specified by the nucleolus for v.

3. The core satisfies all the axioms except CST. Indeed, let us consider two versions of Owen (1972)'s banker game: $N^1 = N^2 := \{1, 2, 3\}$,

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[5] The NTU value doesn't satisfy CD.

[6] That is: $\sigma(v + w) = \sigma(v) + \sigma(w)$, for each pair (v, w) of TU-games.

[7] A value σ is *translation covariant* if $\sigma((v(S) + \sum_{i \in S} x_i)_{S \in P(N)}) = \sigma(v) + x$, for each $x \in \mathbb{R}^N$. This condition imposes σ to vary in covariance with translations of the origin in the utility space. Both the Shapley TU-value and the nucleolus are translation covariant.

$V^1(\{i\}) = V^2(\{i\}) := \{x \in \mathbb{R}^{\{i\}} \mid x_i \leq 0\}$ (for each $i \in \{1,2,3\}$), $V^1(\{1,2\}) := \{x \in \mathbb{R}^{\{1,2\}} \mid x_1 + 4x_2 \leq 100, x_1 \leq 100, x_2 \leq 25\}$, $V^2(\{1,2\}) := \{x \in \mathbb{R}^{\{1,2\}} \mid 4x_1 + x_2 \leq 100, x_1 \leq 25, x_2 \leq 100\}$, $V^1(\{1,3\}) = V^2(\{1,3\}) := \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 \leq 0, x_3 \leq 0\}$, $V^1(\{2,3\}) = V^2(\{2,3\}) := \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 \leq 0, x_3 \leq 0\}$, and $V^1(\{1,2,3\}) = V^2(\{1,2,3\}) := \{x \in \mathbb{R}^{\{1,2,3\}} \mid x_1 + x_2 + x_3 \leq 100\}$. In V^1 (resp. V^2), player 1 (resp. 2) can get 100\$ with the help of player 2 (resp. 1). Player 1 (resp. 2) can send money to player 2 (resp. 1) in exchange of his cooperation, but 3/4 of the total amount of the money sent is lost without the help of player 3 (the banker). When the three players cooperate, any split of the 100\$ created through the cooperation of players 1 and 2 is feasible, thanks to the banker (players 1 and 2 could decide to give some money to player 3 in exchange of his services). The allocation $(20, 20, 60)$ is in the core of both V^1 and V^2. Moreover, it is optimal in $\frac{V^1(N)+V^2(N)}{2}$. Nevertheless, it is not in the core of $\frac{V^1+V^2}{2}$ (it is blocked by coalition $\{1,2\}$ which can achieve the allocation $(50, 50, 0)$).

4. In fact, we proved that the inner core is the minimal (resp. maximal) solution that satisfies EXT, SC and CD (resp. CI, EFF, EXT, SC and CST). As the core satisfies EXT, SC and CD, we conclude again that the inner core is included in the core.

5. The inner core may violate CST when we look at games V for which $V(N)$ is not necessarily smooth. Let us consider the game $\hat{V}^1$ defined by: $\hat{V}^1(\{1\}) := \{x \in \mathbb{R}^{\{1\}} \mid x_1 \leq 10\}$, $\hat{V}^1(\{2\}) := \{x \in \mathbb{R}^{\{2\}} \mid x_2 \leq 10\}$, $\hat{V}^1(\{3\}) := \{x \in \mathbb{R}^{\{3\}} \mid x_3 \leq -20\}$, $\hat{V}^1(\{1,2\}) := \{x \in \mathbb{R}^{\{1,2\}} \mid x_1 + 4x_2 \leq 150, x_1 \leq 110, x_2 \leq 35\}$, $\hat{V}^1(\{1,3\}) := \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 \leq 10, x_3 \leq -20\}$, $\hat{V}^1(\{2,3\}) := \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 \leq 10, x_3 \leq -20\}$, and $\hat{V}^1(\{1,2,3\}) := \{x \in \mathbb{R}^{\{1,2,3\}} \mid x \leq (50, 50, 0)\}$. The game $\hat{V}^1$ is obtained from V^1 by adding 10 to the utilities of both players 1 and 2, by subtracting 20 to the utility of player 3, and by restricting the set of feasible allocations for the grand coalition to the set of vectors that are weakly Pareto dominated by $(50, 50, 0)$. It is easy to check that $(50, 50, 0) \in \left[IC(V^1) \cap IC(\hat{V}^1) \cap \partial\left(\frac{V^1(N)+\hat{V}^1(N)}{2}\right)\right] \setminus IC\left(\frac{V^1+\hat{V}^1}{2}\right)$.

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