Insider Trading with a Random Deadline[†]

René Caldentey
Stern School of Business, New York University,
44 West Fourth Street, Suite 8-77, New York, NY 10012.

Ennio Stacchetti
Department of Economics, New York University,
19 West Fourth Street, 6th floor, New York, NY 10012.

April 24, 2008

Abstract

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider continuously observes a signal that tracks the evolution of the asset's fundamental value. The value of the asset is publicly revealed at a random time. The equilibrium has two regimes separated by an endogenously determined time T. In $[0, T)$, the insider gradually transfers her information to the market. By time T all her information has been transferred and the price agrees with the market value of the asset. After T, the insider trades large volumes and reveals her information immediately, so market prices track the market value perfectly. Despite this market efficiency, the insider is able to collect strictly positive rents after T.

[†] We gratefully acknowledge the feedback of David Pearce. We also thanks Markus Brunnermeier, Lasse Pedersen, and Debraj Ray and seminar participants at NYU.

1 Introduction

This paper studies a model of strategic trading with asymmetric information of an asset whose value follows a Brownian motion. An insider receives a flow of (noisy) signals that tracks the evolution of the asset value. Other traders receive no signals and can only observe the total volume of trade. There is uncertainty about the value of the asset before the insider gets the first signal, hence the first signal generates a lumpy informational asymmetry between the insider and the rest of the market participants. The signals the insider receives later are equally informative, but they contribute only marginally to the informational asymmetry. The information advantage continues until an unpredictable time when a public announcement reveals the current value of the asset to all the traders.

Kyle (1985) introduced a dynamic model of insider trading where an insider receives only one signal and the fundamental asset value does not change over time. Through trade, the insider progressively releases her private information to the market as she exploits her informational advantage. The market is also populated by many liquidity traders that are uninformed and trade randomly. At time 0, the insider observes the value of an asset. The same information is publicly released later, at time 1, to all market participants. In each trading period in the time interval $[0, 1]$, traders submit order quantities to a risk-neutral market maker who sets prices competitively and trades in his own account to clear the market. The market maker cannot observe individual trades, but can observe the total volume of trade in each trading period. The market maker also knows (in equilibrium) the strategy of the informed trader, and sets prices efficiently conditional on past and present volumes of trade.

Kyle constructs a linear equilibrium where in each period the price adjustment is proportional to the volume of trade, and the insider's volume of trade is proportional to the gap between the asset value and the current market price. The market maker's estimate of the asset value, reflected in the current market price, improves over time. As the public announcement date approaches, this estimate converges to the value of the asset and the insider trades frantically in her desire to exploit any price differential.

Our model differs from Kyle's model in three important ways. First, the fundamental value of the asset follows a Brownian motion and therefore changes continuously over time. Second, in addition to the initial observation, the insider continuously receives a signal of the current fundamental value of the asset. Third, the public announcement date is unpredictable: it has an exponential distribution.

The first difference by itself is irrelevant. In Kyle's model it makes no difference whether at time 0 the insider observes the true value of the asset or just an unbiased signal. Moreover, the model where the insider observes the true value and the value of the asset follows a Brownian motion is formally equivalent to a model where the initial observation is an unbiased signal of the final value of the asset. But this feature of our model becomes important when it is combined with the second feature. Finally, the third feature removes the pressure in Kyle's model behind the trade frenzy that occurs as the announcement date approaches. In our model, where the announcement date is not deterministic, the insider has no urgency to exhaust all arbitrage opportunities, and release all her private information in the process, by a particular deadline. Thus, while it is evident that in Kyle's model the price will become efficient (in the sense that it incorporates all the available information) as time reaches the announcement date, it is unclear whether in our model the insider will ever fully reveal her private information.

It is exactly this feature of the equilibrium of the fixed horizon model that Back (1992) exploits to develop his elegant "backward programming" solution method. In a model with a random horizon, Back's method is not directly applicable.

Our model is not the first to introduce a public announcement with random time. Back and Baruch (2004) compare the models of Kyle (1985) and Glosten and Milgrom (1985). To facilitate the comparison, they adopt a Glosten and Milgrom model with a single long-lived insider (who times her transactions strategically) and a Kyle model with a random terminal time and a risky asset that takes only the values 0 or 1.

Kyle's original model is in discrete time. The insider and liquidity traders place their orders at the beginning of each period and the market maker sets the price *after* observing the total volume of trade. Kyle then shows that as the period length Δ converges to 0, the equilibrium converges to an equilibrium of the continuous-time limit model in which the agents can trade continuously and the market maker adjusts prices continuously. He then interprets the continuous-time model as a good representation of a discrete-time model where the agents can trade frequently. We maintain this interpretation and view the continuous-time model as a mathematical convenience that affords us the powerful tools of stochastic calculus. The discrete-time version of our model has a unique equilibrium that converges to a well defined strategy profile as $\Delta \downarrow 0$. However, in our case, this limit strategy is not an equilibrium of the continuous-time model. The interpretation of the continuous-time model is therefore delicate and needs to be examined more carefully[1]. The lack of "continuity" arises because in the limit the insider wants to trade at infinite rates after some time T. This allows the insider to collect positive rents even though the price perfectly tracks the

---

[1] A recent paper by Fudenberg and Levine (2007) studies the limit of equilibria of infinitely repeated games as $\Delta \downarrow 0$. They show that in general the limit of equilibria of discrete-time games does not coincide with the equilibria of the limit continuous-time game.

value of the asset after this time. However, after T, the insider's payoff function evaluated at the limit strategy is 0. Therefore, as we explain in Section 4, the limit strategy cannot be an equilibrium of the continuous-time model.

To deal with this discontinuity while still using stochastic calculus, we introduce a sequence of continuous-time models with an arbitrary upper bound on the transaction rates of the insider. As the upper bound is relaxed, the equilibrium of the constrained continuous-time model converges to the same limit strategy of the discrete-time model. Thus, we interpret this limit as the appropriate "equilibrium" of the continuous-time model, even though this strategy profile does not satisfy the standard conditions for an equilibrium.

Our model includes various special cases. The value of the asset remains constant over time if the variance of its Brownian motion is reduced to 0. Since in our model the insider observes the initial value without noise, the signals that track the value of the asset over time become superfluous. This version of our model is similar to Kyle's model, where the insider is endowed only with an initial piece of private information, but with a random end time. Alternatively, we can specialize our model to give the insider no initial informational advantage. This is accomplished by informing *all* traders of the initial value of the asset. In this version of the model, the insider's informational advantage arises exclusively from her ability to observe the evolution of the asset value. This is an important model in its own right. An interesting question in this model is how the insider 'manages' the information asymmetry. For example, the insider could let the information asymmetry (the variance of the uninformed traders' estimate of the current value) grow to reach asymptotically a certain limit or without bound. The larger is the information asymmetry, the more likely it is that the market will substantially misprice the asset, and therefore, the larger are the profitable arbitrage opportunities. Thus, in this model as well it is not evident how much of the insider's information is incorporated in the market price and how quickly this happens. We study this special case in the process of constructing an equilibrium for our general model. It turns out that in equilibrium the insider fully reveals her information as soon as she receives it. Hence, the market price equals the asset value at all times. Yet, the insider makes strictly positive profits. In independent work, Chau and Vayanos (2006) reach the same conclusion (for this case without initial informational asymmetry) in a slightly different model. They assume that the insider receives a flow of information, the asset pays a dividend, and there is no public announcement. In addition, they assume that the market maker continuously observes a noisy signal of the value of the asset. In the absence of this noisy signal, their model would be formally equivalent to ours. Chau and Vayanos (2006) limit attention to the steady state of their model and do not study how the equilibrium approaches the steady state. One implication of our results is that in the absence of an initial information asymmetry, the steady state is reached 'immediately' (as the period length goes to 0), so although Chau and Vayanos (2006) assume that trading has been taking place indefinitely, this is not needed.

The equilibrium of our general model has a striking feature. There is a time T, endogenously determined in equilibrium, by which the insider reveals all her information (if the public announcement has not yet occurred). Thus, even though there is no deterministic deadline, the price converges to the asset value at time T. Moreover, time T divides the equilibrium into two phases. As long as the public announcement does not occur, in the interval $[0, T)$ the insider gradually transfers her information to the market and the market's uncertainty about the value of the asset decreases to 0 monotonically. In the interval $[T, \infty)$, the insider trades large volumes and reveals her information immediately, so market prices track the asset value perfectly. Nevertheless, as we explained above, after T the insider collects strictly positive rents. In $[0, T)$ the insider is indifferent about her order quantities, though she trades according to a deterministic function of the current price and value of the asset. Therefore, she is indifferent about purchasing an additional share of the asset now or in the future, even though she discounts future payoffs. This is so because the market compensates her more generously in the future for any price differential. In $[T, \infty)$, her compensation, as a function of the price differential, is constant over time, and thus she is eager to cash in her rents as soon as arbitrage opportunities materialize.

We conclude the Introduction with a quick review of the vast literature on insider trading.[2] Two of the most influential papers in the area of strategic trading with asymmetric information are Kyle (1985) and Glosten and Milgrom (1985). These classic papers formalize the intuitive story of Bagehot (1971) that the market provides a mechanism to compensate informed traders for their superior information, while liquidity traders are willing to make (small) losses for the benefit of carrying out their transactions immediately.[3] The literature that builds upon Kyle (1985) is more closely related to our work. In a continuous-time setting, Back (1992) considers a general distribution for the insider's private signal (Kyle assumes a normal distribution) and proves the existence and *uniqueness* of an equilibrium pricing rule. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1996) consider a market with multiple competing insiders. They show that competition among insiders

---

[2] For a comprehensive review of this literature, and its connection to the broader market microstructure theory, we refer the reader to O'Hara (1997), Brunnermeier (2001), Biais et al. (2005), Amihud et al. (2006) and references therein.

[3] Three notable extensions of the Glosten and Milgrom model are Easley and O'Hara (1987) that study the impact of block trading on the bid-ask spread, Glosten (1989) that considers a monopolist specialist that maximizes expected profits, and Dasgupta and Prat (2005) that analyze a model where some insiders receive superior signals and informed traders care about their reputations.

accelerate the release of their private information. In a one-period model with heterogeneous insiders, Spiegel and Subrahmanyam (1992) replace Kyle's uninformed liquidity traders (and their exogenous price-inelastic noisy trades) with strategic utility-maximizing agents trading for hedging purposes. In a multi-period setting, Mendelson and Tunca (2004) propose an alternative endogenous liquidity trading model allowing for various type of market information. In contrast to Kyle's model, Mendelson and Tunca assume that the insider's private information acquisition is costly. Similar to our model, Back and Pedersen (1998) consider the case where the insider continuously observes private information, which evolves as a Gaussian martingale over a fixed time horizon. They assume that the insider's initial amount of private information is sufficiently high in order to prove that an equilibrium exists. We show that a similar condition is required in our setting with a random announcement date. Furthermore, we also show that it is precisely when this condition is violated –*i.e.*, when the insider's initial private information is small compared to the inflow on new information– that our equilibrium reaches market efficiency at a fixed time T and preserves it thereafter.

The rest of the paper is organized as follows. Section 2 introduces the continuous-time model. In Section 3 we construct an equilibrium for a constrained model where the insider's trading rate is exogenously bounded. Section 4 discusses the limit of the constrained equilibrium as the bound on the insider's trading rate grows arbitrarily large. We also show that this limit strategy is *not* an equilibrium of the unconstrained model presented in Section 2. In Section 5 we prove, however, that this limit strategy provides a good approximation to a discrete-time model in which the agents act frequently. Section 6 includes our concluding remarks.

2 Model Description

The market participants are the insider, the market maker and a (large) number of liquidity traders. The insider (and only she) continuously receives private information about the fundamental value of the asset. The insider and the liquidity traders adjust their net holdings of the asset and the market maker sets the price at which these trades are executed continuously. This trading process terminates at an unpredictable random time τ when the fundamental value of the asset becomes public knowledge. At this time, the market price immediately matches the fundamental value and the insider loses her informational advantage.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with two independent standard Brownian motions B_t^v and B_t^y, where $t \in [0, \infty)$ denotes (calendar) time. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the usual filtration generated by (B^v, B^y). The value of the fundamental at time t is $\bar{V}_t$, which we assume evolves over time as an arithmetic Brownian motion

$$\mathrm{d}\bar{V}_t = \bar{\sigma}_v \, \mathrm{d}B_t^v,$$

for some constant $\bar{\sigma}_v \geq 0$. The initial value $\bar{V}_0$ is drawn from a normal distribution with mean $\bar{v}_0$ and variance $\bar{\Sigma}_0$. The insider alone observes the (stochastic) evolution of $\bar{V}_t$ during $t \in [0, \tau)$. The market maker and the rest of the market participants only know the distribution of $\bar{V}_0$. The random time τ when the value of the fundamental becomes public knowledge is exponentially distributed with mean $1/\theta$, and is independent of $\mathbb{F}$.

Liquidity traders are not strategic agents and they are motivated to trade for idiosyncratic reasons. They trade so as to match a moving target for their net holding of the asset. Their holding target Y_t at time t follows an arithmetic Brownian motion

$$\mathrm{d}Y_t = \sigma_y \, \mathrm{d}B_t^y$$

for some constant $\sigma_y > 0$. We assume that the insider and the market maker know $\bar{\sigma}_v$ and σ_y. We denote by X_t the insider's cumulative orders (net position on the asset) at time t and by $Z_t = X_t + Y_t$ the total (net) volume of trade up to time t. Without loss of generality, hereafter we assume $X_0 = 0$ and $Y_0 = 0$.

The market maker only observes the process Z and cannot distinguish between liquidity and insider trading. As a result, the pricing rule $\{P_t\}$ that he selects is adapted to $\mathbb{F}^M = \{\mathcal{F}_t^M\}$, the σ-field generated by Z_t. On the other hand, the insider trading strategy X_t is only required to be adapted to the finer filtration $\mathbb{F}$ reflecting her informational advantage[4].

For a given pair (X, P), the insider's expected discounted payoff $\Pi(P, X)$ is defined as the difference between the discounted market value of her portfolio at time τ minus the discounted cost required to build this portfolio during $[0, \tau)$. That is,

$$\Pi(P, X) = \mathbb{E}\left[e^{-\delta\tau} \, \bar{V}_\tau \, X_\tau - \int_0^\tau e^{-\delta t} P_t \, \mathrm{d}X_t - \int_0^\tau e^{-\delta t} \mathrm{d}[X, P]_t \right],$$

where $\delta > 0$ is the discount factor and $[X, P]_t$ is the quadratic covariation between X_t and P_t. Intuitively, this term arises because the price paid by the insider is computed 'at the end of the trading period', and therefore it includes the effect of the insider's 'last trade' $\mathrm{d}X_t$.[5] As a technical remark, to ensure that the insider's payoff is well defined (in particular

---

[4] By letting $X_t \in \mathcal{F}_t$ we are assuming that the insider observes Z_t. It may be more natural to assume that the insider observes the price process P_t rather than Z_t. However, in equilibrium the pricing rule is a monotonic function of Z_t that the insider can invert.

[5] In other words, to marginally change her portfolio from X_t to $X_t + \mathrm{d}X_t$ the insider incurs an incremental cost $(P_t + \mathrm{d}P_t) \, \mathrm{d}X_t = P_t \, \mathrm{d}X_t + \mathrm{d}[X, P]_t$. For a formal derivation, see equation (11) in Back (1992).

the stochastic integral with respect to X_t) we restrict the process X_t to the class $\mathcal{S}$ of semimartingales adapted to $\mathcal{F}_t$.

For the analysis that follows, we find it convenient to define the *intrinsic value* V_t as the expected discounted value of the fundamental at time τ given the insider's information at time t. That is,

$$V_t = \mathbb{E}[e^{-\delta(\tau-t)}\,\bar{V}_\tau \mid \mathcal{F}_t, t < \tau] = \frac{\theta}{\theta+\delta}\,\bar{V}_t.$$

We also define $\sigma_v = \bar{\sigma}_v\,\theta/(\theta+\delta)$ so that V_t is a driftless Brownian motion with dynamics

$$\mathrm{d}V_t = \sigma_v\,\mathrm{d}B_t^v.$$

Let us also rewrite the insider's payoff using the following identity

$$e^{-\delta\,\tau}\,\bar{V}_\tau\,X_\tau = \int_0^\tau e^{-\delta\,\tau}\,\bar{V}_t\,\mathrm{d}X_t + \int_0^\tau e^{-\delta\,\tau}\,X_t\,\mathrm{d}\bar{V}_t + \int_0^\tau e^{-\delta\,\tau}\,\mathrm{d}[X,\bar{V}]_t,$$

where $[X,\bar{V}]_t$ is the quadratic covariation between X_t and $\bar{V}_t$. Plugging this identity back in Π, taking expectation and canceling the stochastic integral with respect to the martingale $\bar{V}_t$, we get

$$\begin{aligned}\Pi(P,X) &= \mathbb{E}\left[\int_0^\tau \left(e^{-\delta\,\tau}\,\bar{V}_t - e^{-\delta\,t}\,P_t\right)\mathrm{d}X_t + \int_0^\tau e^{-\delta\,t}\mathrm{d}[X,V]_t - \int_0^\tau e^{-\delta\,t}\mathrm{d}[X,P]_t\right]\\ &= \mathbb{E}\left[\int_0^\infty e^{-\mu\,t}(V_t - P_t)\,\mathrm{d}X_t + \int_0^\infty e^{-\mu\,t}\mathrm{d}[X,V]_t - \int_0^\infty e^{-\mu\,t}\mathrm{d}[X,P]_t\right],\end{aligned}$$

where $\mu = \delta + \theta$. Note that the second equality is based on the fact that τ is exponentially distributed with rate θ and is independent of $\mathcal{F}_t$. We are now ready to define the notion of a market equilibrium.

Definition 1 (Market Equilibrium) *A strategy for the market maker is an* $\mathcal{F}_t^M$*-adapted process* $P = \{P_t\}_{0\le t\le\tau}$*, and a strategy for the insider is an* $\mathcal{F}_t$*-adapted process* $X = \{X_t\}_{0\le t\le\tau}$ *such that* $X \in \mathcal{S}$*. The profile* (P,X) *is an equilibrium if (i) for any* $0 \le t < \tau$

$$P_t = \mathbb{E}[V_t \mid \mathcal{F}_t^M, X],$$

and (ii) given P*,* $\Pi(P,X)$ *is bounded and* $\{X_t\}$ *maximizes* $\Pi(P,X)$*.*

One might think that condition (i) alone would guarantee that the market maker's expected payoff is 0. However, it is possible to construct profiles (P,X) satisfying condition (i) for which the market maker makes infinite losses. The condition $\Pi(P,X) < \infty$ together with (i) does ensure that the market maker breaks even in expectation.

The model is not exactly a game and our definition of an equilibrium does not coincide with that of a Nash equilibrium. However, Kyle (1985) suggests that this definition would coincide with that of a Nash equilibrium in a game where two market makers simultaneously bid prices after observing the current volume of trade and the winner gets the right to clear the market at the wining price [6]. In equilibrium, this competition drives the market maker to set the price at time t equal to the expected value of the asset's market value given the history of information he has observed so far and the insider's trading strategy. The market maker only uses his history to make inferences about the past choices of the insider and therefore, indirectly, about the distribution of V_t.

In equilibrium, P_t evolves as a martingale with respect to the market's public information $\mathcal{F}_t^M$. Intuitively, this property is consistent with the fact that any predictable drift on the price dynamics would be instantaneously exploited by the insider. By a similar token, one can argue that in equilibrium the market net holding, Z_t, must also evolve as a martingale with respect to $\mathcal{F}_t^M$. Indeed, given the martingale nature of liquidity trading, a systematic trend on Z_t could only be interpreted as the result of insider trading driven by a mispriced asset. Below, we will characterize an equilibrium that is consistent with this intuition. A useful implication of both Z_t and P_t being martingales with respect to $\mathcal{F}_t^M$ is that we can connect the stochastic evolution of these two processes in a rather simple way. Indeed, under some additional technical conditions[7], we can apply the *martingale representation theorem* to argue that there exists an $\mathcal{F}_t^M$-adapted process λ_t such that for all $t < \tau$

$$\mathrm{d}P_t = \lambda_t \,\mathrm{d}Z_t.$$

That is, price changes are locally proportional to the volume of trade.

Turning to the insider's trading strategy X_t we see that in equilibrium she chooses her strategy so as to maximize her expected discounted profit, given that she knows how the market maker will choose prices. In Definition 1, X_t is only required to be an $\mathcal{F}_t$-adapted semimartingale. In equilibrium, however, one expects X_t to satisfy additional regularity conditions. For instance, it seems natural to rule out discontinuities in X_t because they would immediately inform the market maker that he is mispricing the asset. (Recall that liquidity trading is driven by a continuous process). Hence, in equilibrium we expect X_t to

---

[6] To avoid collusion, we can assume that there is a large population of market makers and that each market maker participates in the bidding game only once.

[7] Such as requiring P_t to be a Markov process or requiring that $P_t = G(t, Z_t)$ for a deterministic function G. This latter condition is commonly used in the literature to formalize the notion of a Markovian equilibrium, *e.g.*, Back (1992).

be a continuous semimartingale adapted to $\mathcal{F}_t$ and to admit a decomposition

$$\mathrm{d}X_t = \xi_t \,\mathrm{d}t + \phi_t \,\mathrm{d}B_t^v + \psi_t \,\mathrm{d}B_t^y,$$

for three $\mathcal{F}_t$-adapted processes ξ, ϕ and ψ.

In the standard insider trading model (*e.g.*, Kyle 1985 or Back 1992), the fundamental value is constant and this value is is publicly revealed at a fixed time. Under these assumptions, Back (1992) proves (see Lemma 2) that an optimal trading strategy satisfies $\phi = \psi = 0$ and so in his setting there is no loss in generality in restricting attention to trading strategies X_t that are absolutely continuous with respect to time. As a result, the insider's trading strategy is safely hidden by liquidity trading[8] and her private information is never fully revealed to the market prior to the announcement time.

Despite the similarities, Back's method is not directly applicable to our setting with random termination time and a stochastic asset value. As we will see below, these variations lead to a fundamental change in the way the insider manages her informational rents. In particular, the unpredictability of the announcement date leads the insider to exhaust all her information stock by a fixed time T and then exploit instantaneously any new piece of information afterwards. Hence, in our model (for some range of parameters), the market price becomes strongly efficient before the public announcement. Efficiency is maintained after T because the insider uses a trading strategy X_t with unbounded variation that fully offsets the price impact of liquidity trading.

To formalize these ideas, we restrict attention to a particular class of Markovian equilibria.

Definition 2 (Markovian Profile) *We say that a strategy profile* (X, P) *such that*

$$\mathrm{d}X_t = \xi_t \mathrm{d}t + \phi_t \,\mathrm{d}B_t^v + \psi_t \,\mathrm{d}B_t^y \qquad \textit{and} \qquad \mathrm{d}P_t = \lambda_t \,\mathrm{d}Z_t, \qquad t < \tau$$

is Markovian *if* ξ_t*,* ϕ_t *and* ψ_t *are deterministic functions of* (t, Z_t, P_t, X_t, V_t) *and* λ_t *is a deterministic function of* (t, Z_t, P_t)*.*

3 Equilibrium with Restricted Trading

In this section, we construct a Markovian equilibrium for a constrained model in which the insider trading strategy X_t is a process of bounded variation. Besides being a natural

---

[8] Roughly speaking, the increments of the liquidity trading process are order $\sqrt{\mathrm{d}t}$ while the increments of the insider trading strategy are order $\mathrm{d}t$.

constraint in practice (*i.e.*, transactions costs would be prohibitively high if X_t were of unbounded variation), we impose this restriction to capture frictions that are naturally present in a discrete-time model. As we discuss later in Section 5, we view the continuous-time model as an appropriate representation for the dynamics of a market in which trading occurs at discrete yet frequent time epochs.

Given $\bar{\beta} > 0$, we now restrict the insider's strategy space to the set $\mathcal{C}(\bar{\beta})$ of all processes X such that

$$\mathrm{d}X_t = \beta_t \, (V_t - P_t) \, \mathrm{d}t$$

where β_t is deterministic function of (t, Z_t, P_t, X_t, V_t) such $|\beta_t| \leq \bar{\beta}$ for all $t \geq 0$. Hence, $\bar{\beta}$ is the insider's maximal transaction rate (per dollar of price gap). Obviously, if $X \in \mathcal{C}(\bar{\beta})$, then X is of bounded variation.

The continuous-time model with this constrained strategy space does have an equilibrium. When X is of bounded variation, the quadratic covariations $[X, V]_t$ and $[X, P]_t$ are both zero, and accordingly the last two terms of $\Pi(P, X)$ drop out and

$$\Pi(P, X) = \mathbb{E}\left[\int_0^\infty e^{-\mu \, t} \, \beta_t \, (V_t - P_t)^2 \, \mathrm{d}t\right].$$

Let us suppose that the market maker uses the pricing rule

$$\mathrm{d}P_t = \lambda_t \, \mathrm{d}Z_t, \qquad t \geq 0, \tag{1}$$

for some nonnegative process λ_t. Then, under the restriction $X \in \mathcal{C}(\bar{\beta})$, the evolution of P_t is governed by the following SDE

$$\mathrm{d}P_t = \lambda_t \, [\beta_t (V_t - P_t) \mathrm{d}t + \sigma_y \, \mathrm{d}B_t^y].$$

Note that both the insider's payoff function and the dynamics of P_t depend on P_t and V_t only through their difference. Thus, we find it convenient to define the price-gap process $M_t = V_t - P_t$ with dynamics

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_v \, \mathrm{d}B_t^v - \lambda_t \, \sigma_y \, \mathrm{d}B_t^y, \qquad t \geq 0. \tag{2}$$

The process $\sigma_v \, B_t^v - \lambda_t \, \sigma_y \, B_t^y$ is a driftless Gaussian process with variance $\sigma_t^2 = \sigma_v^2 + \lambda_t^2 \, \sigma_y^2$. Therefore,

$$\mathrm{d}M_t = -\lambda_t \, \beta_t \, M_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t \qquad t \geq 0,$$

where B_t is a standard Brownian motion. In equilibrium both λ_t and β_t are nonnegative and so the process M_t reverts towards 0. We define the value function

$$\begin{aligned} \Pi(t, M) \quad &= \quad \sup_{|\beta_t| \leq \bar{\beta}} \; \mathbb{E}\left[\int_t^\infty e^{-\mu \, (s-t)} \, \beta_s \, M_s^2 \, \mathrm{d}s\right] \\ &\quad \text{s.t.} \quad \mathrm{d}M_s = -\lambda_s \, \beta_s \, M_s \, \mathrm{d}t + \sigma_s \, \mathrm{d}B_s \quad \text{for } s \geq t, \quad \text{and} \quad M_t = M. \end{aligned} \tag{3}$$

The process $\Pi(t, M)$ is the insider's optimal expected profit-to-go starting at time t with an initial price-value gap $M_t = M$. We note that M_t and Π depend on both the pricing policy λ and $\bar{\beta}$. When we wish to emphasize this dependence we will include λ and/or $\bar{\beta}$ as part of their arguments, for example, we will write $M_t(\bar{\beta})$ or $\Pi(t, M, \lambda, \bar{\beta})$.

The dynamic programming HJB equation for $\Pi(t, M)$ is

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ -\lambda_t \, \beta \, M \, \Pi_M + \frac{1}{2} \, (\sigma_v^2 + \lambda_t^2 \, \sigma_y^2) \, \Pi_{MM} + \Pi_t - \mu \, \Pi + M^2 \, \beta \right\}, \qquad t \in [0, \infty)$$

where Π_M (Π_{MM}) and Π_t are the first (second order) partial derivative of Π with respect to M and t, respectively. We will show that the profit-to-go is a quadratic function $\Pi(t, M_t) = \alpha_t \, M_t^2 + \gamma_t$, where α_t and γ_t are two deterministic functions of t. Then, the HJB reduces to

$$0 = \max_{|\beta| \leq \bar{\beta}} \left\{ \left[\beta \, (1 - 2\lambda_t \, \alpha_t) + \dot{\alpha}_t - \mu \, \alpha_t \right] M^2 + \alpha_t (\sigma_v^2 + \lambda_t^2 \sigma_y^2) + \dot{\gamma}_t - \mu \gamma_t \right\}. \tag{4}$$

The HJB is linear on β, so unless $2\lambda_t \, \alpha_t = 1$, the insider uses a "bang-bang" strategy trading at a maximum rate $|\beta_t| = \bar{\beta}$. When $2\lambda_t \, \alpha_t = 1$, she is indifferent in her choice of β_t. The market maker's equilibrium condition is characterized in the following proposition.

Proposition 1 *Suppose the insider selects a deterministic trading rate β_t, and the market maker chooses the pricing rule (1). Then the market maker's equilibrium condition $P_t = \mathbb{E}[V_t | \mathcal{F}_t^M, X]$ is satisfied if λ_t is a deterministic function such that*

$$\Sigma_t \, \beta_t = \sigma_y^2 \, \lambda_t \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2 \, \lambda_t^2, \tag{5}$$

where $\Sigma_t = \mathbb{E}[(V_t - P_t)^2 | \mathcal{F}_t^M, X]$ and $\dot{\Sigma}_t$ represents its first derivative with respect to t.

Note that if $\lambda_t < 0$, choosing $\beta_t = \bar{\beta} > 0$ is optimal because it maximizes current payoffs and *increases* the gap between V_t and P_t. For example, if $V_t - P_t > 0$ the insider maximizes the flow of profits when $\beta_t = \bar{\beta}$, while at the same time she *decreases* the price, and hence *increases* the profits she makes in future purchases. However, by Proposition 1 and the fact that $\Sigma_t \geq 0$, λ_t and β_t must have the same sign. Since $\beta_t = \bar{\beta} > 0$, λ_t should also be positive, contradicting our assumption. Thus, in equilibrium, $\lambda_t \geq 0$ for all t.

In the following theorem we construct an equilibrium which is defined by a pair of deterministic nonnegative processes (λ_t, β_t) that satisfy (5) and solve (3) for $t = 0$. The statement of the theorem requires some addition notation. Let $L = 2\sigma_v \bar{\beta} / \sigma_y$. Define the auxiliary function

$$h(x) = \int_0^\infty e^{-(1 + \frac{\mu}{L}) \, u} \, \frac{1 - x^2}{(1 - x \, e^{-u})^2} \, \mathrm{d}u$$

and denote by $\tilde{k} = \min\{x \geq 0 \mid h(x) = 1\}$. One can show that $\tilde{k}$ exists and is strictly less than 1. We also define

$$\widetilde{\Sigma} = \frac{\sigma_v\,\sigma_y}{\bar{\beta}} \left[\frac{1+\tilde{k}}{1-\tilde{k}}\right].$$

Theorem 1 *An equilibrium of the game with upper bound $\bar{\beta}$ is characterized by one of the following two cases.*

Case I*: If $\Sigma_0 > \widetilde{\Sigma}$ then there exist nonnegative constants λ_0 and T such that*

$$\lambda_t = \begin{cases} \lambda_0\, e^{-\mu\, t} & \text{if} \quad t < T \\ \dfrac{\sigma_v}{\sigma_y}\left[\dfrac{e^{L\,t} + k\, e^{L\,T}}{e^{L\,t} - k\, e^{L\,T}}\right] & \text{if} \quad t \geq T \end{cases} \qquad \text{and} \qquad \beta_t = \begin{cases} \dfrac{\sigma_y^2 \lambda_t}{\Sigma_t} & \text{if} \quad t < T \\ \bar{\beta} & \text{if} \quad t \geq T, \end{cases} \tag{6}$$

where

$$\Sigma_t = \begin{cases} \Sigma_0 + \sigma_v^2 t - \dfrac{\lambda_0^2\,\sigma_y^2}{2\mu}\left[1 - e^{-2\,\mu\,t}\right] & \text{if} \quad t < T \\ \dfrac{\sigma_v\,\sigma_y}{\bar{\beta}}\left[\dfrac{e^{L\,t} + k\, e^{L\,T}}{e^{L\,t} - k\, e^{L\,T}}\right] & \text{if} \quad t \geq T, \end{cases} \tag{7}$$

and $k = \tilde{k}$. Moreover, in equilibrium, $\Pi(t, M) = \alpha_t\, M^2 + \gamma_t$, where

$$\alpha_t = \begin{cases} \dfrac{e^{\mu\,t}}{2\,\lambda_0} & \text{if} \quad t < T \\ \bar{\beta}\, e^{-(L-\mu)\,t}\left[e^{L\,t} - k\, e^{L\,T}\right]^2 \displaystyle\int_t^\infty e^{(L-\mu)\,s}\left[e^{L\,s} - k\, e^{L\,T}\right]^{-2} \mathrm{d}s & \text{if} \quad t \geq T \end{cases} \tag{8}$$

$$\gamma_t = \begin{cases} \left[\gamma_0 - \dfrac{\lambda_0 \sigma_y^2}{4\mu} - \dfrac{\sigma_v^2}{2\lambda_0} t\right] e^{\mu t} + \dfrac{\lambda_0\sigma_y^2}{4\mu} e^{-\mu t} & \text{if} \quad t < T \\ \bar{\beta} e^{\mu\,t} \displaystyle\int_t^\infty e^{(L-\mu)s}\left[e^{L\,s} - k\, e^{L\,T}\right]^{-2} \int_t^s \sigma_u^2 e^{-L\,u}\left[e^{L\,u} - k\, e^{L\,T}\right]^2 \mathrm{d}u\, \mathrm{d}s & \text{if} \quad t \geq T. \end{cases} \tag{9}$$

The constants (λ_0, γ_0, T) are determined by the value-matching conditions

$$\lim_{t \uparrow T} \lambda_t = \lambda_T, \qquad \lim_{t \uparrow T} \Sigma_t = \Sigma_T \quad \text{and} \quad \lim_{t \uparrow T} \gamma_t = \gamma_T. \tag{10}$$

Though the value $\Pi(t, M)$ is defined piecewise, it is continuously differentiable in t and twice continuously differentiable in M.

Case II*: If $\Sigma_0 \leq \widetilde{\Sigma}$ then an equilibrium is defined by (6), (7), (8) and (9) with $T = 0$ and*

$$k = \frac{\bar{\beta}\,\Sigma_0 - \sigma_y\,\sigma_v}{\bar{\beta}\,\Sigma_0 + \sigma_y\,\sigma_v}.$$

Let $\Sigma_\infty = \sigma_y\,\sigma_v/\bar\beta$. In equilibrium, Σ_t is monotone for all $\Sigma_0 \neq \Sigma_\infty$: strictly increasing if $0 \le \Sigma_0 < \Sigma_\infty$ and strictly decreasing if $\Sigma_0 > \Sigma_\infty$ (and $\Sigma_t \equiv \Sigma_\infty$ when $\Sigma_0 = \Sigma_\infty$). Moreover, $\Sigma_t \to \Sigma_\infty$ as $t \to \infty$. The equilibrium is truly Markovian with state variable Σ_t, so in fact $\lambda_t = \lambda(\Sigma_t)$ and $\beta_t = \beta(\Sigma_t)$ (and similarly for the other variables α_t and γ_t). There are two types of states: for $\Sigma_t \in [0, \widetilde\Sigma]$, $\beta_t = \bar\beta$ and the insider trades as fast as possible, while for $\Sigma_t \in (\widetilde\Sigma, \infty)$, the insider is kept indifferent and trades at intermediate rates. The monotonicity of Σ_t implies that there exists a time T such that $\Sigma_t \in [0, \widetilde\Sigma]$ for all $t \ge T$, and hence after T, the insider always trades as fast as she can. When $\Sigma_t \in [0, \Sigma_\infty)$, Σ_t is monotonically increasing because even though the insider sets $\beta_t = \bar\beta$, the information that she transfers to the market cannot compensate for the volatility of the fundamental value. When $\Sigma_t > \Sigma_\infty$, Σ_t is monotonically decreasing and prices become more accurate over time.

We can define the actions of the insider and the market maker (as well as the auxiliary variables α and γ) in terms of the current state as follows. For a "corner state" $\Sigma_t \in [0, \widetilde\Sigma]$, let

$$\lambda(\Sigma_t) = \frac{\bar\beta}{\sigma_y^2}\,\Sigma_t, \qquad \beta(\Sigma_t) = \bar\beta, \qquad k(\Sigma_t) = \frac{\bar\beta\,\Sigma_t - \sigma_y\,\sigma_v}{\bar\beta\,\Sigma_t + \sigma_y\,\sigma_v}\,, \qquad \text{and}$$
$$\alpha(\Sigma_t) = \bar\beta\,(1-k)^2 \int_0^\infty e^{(L-\mu)s}[e^{L\,s} - k]^{-2}\mathrm{d}s.$$

Here $2\lambda(\Sigma_t)\alpha(\Sigma_t) \le 1$ (with strict inequality if $\Sigma_t < \widetilde\Sigma$), and by the HJB equation (4), $\beta_t = \beta(\Sigma_t) = \bar\beta$ is indeed optimal for the insider. While for $\Sigma_t > \widetilde\Sigma$ we could still define $(k, \lambda(\Sigma_t), \beta(\Sigma_t), \alpha(\Sigma_t))$ as above, $2\lambda(\Sigma_t)\alpha(\Sigma_t) > 1$ and $\bar\beta$ would not be optimal for the insider (the insider's optimal action would be to set $\beta_t = -\bar\beta$). Thus, if $\Sigma_0 > \widetilde\Sigma$, it is not possible to construct an equilibrium where the insider trades as fast as possible all the time. Note that when $\Sigma_t = \widetilde\Sigma$, $k = \tilde k$ and $2\lambda(\Sigma_t)\alpha(\Sigma_t) = 1$.

For an "indifference state" $\Sigma_t > \widetilde\Sigma$, let $(\tilde t, \lambda_t)$ be the solution of the system

$$\Sigma_t + \sigma_v^2 \tilde t - \frac{\lambda_t^2\,\sigma_y^2}{2\mu}\left[1 - e^{-2\,\mu\,\tilde t}\right] = \widetilde\Sigma \qquad \text{and} \qquad \lambda_t\,e^{-\mu\,\tilde t} = \lambda(\widetilde\Sigma).$$

That is, $\tilde t$ is the time it takes (in equilibrium) for the variance to decrease from Σ_t to $\widetilde\Sigma$. It is easy to see that $\tilde t$ and λ_t are increasing in Σ_t. The larger is Σ_t, the more time it takes for the variance to reach $\widetilde\Sigma$ and the more sensitive is the price to the volume of trade. Now let

$$\lambda(\Sigma_t) = \lambda_t, \qquad \beta(\Sigma_t) = \frac{\sigma_y^2\lambda_t}{\Sigma_t}\,, \qquad \text{and} \qquad \alpha(\Sigma_t) = \frac{1}{2\,\lambda_t}.$$

Here, by definition, $2\lambda(\Sigma_t)\alpha(\Sigma_t) = 1$ and the HJB equation (4) implies that the insider is (locally) indifferent. The definition of λ_t above implies that $\lambda(\Sigma_t)$ is continuous at $\Sigma_t = \widetilde\Sigma$. If

$\lambda(\Sigma_t)$ jumped down as $\Sigma_t \downarrow \widetilde{\Sigma}$, the insider would have an incentive to delay her transactions until time T, and if it jumped up, she would have an incentive to act faster before T. The rate $\beta(\Sigma_t)$ is also continuous at $\Sigma_t = \widetilde{\Sigma}$ since $\beta(\Sigma_t) \uparrow \bar{\beta}$ as $\Sigma_t \downarrow \widetilde{\Sigma}$. The continuity of λ also implies that $\alpha(\Sigma_t)$ is continuous at $\Sigma_t = \widetilde{\Sigma}$.

When $\Sigma_0 > \widetilde{\Sigma}$, it is not possible to set $T = \infty$ and make the insider indifferent all the time. If one sets $T = \infty$ in (6)–(9), the resulting value function $\Pi(t, M)$ and trading rate β_t satisfy the HJB equation (4), and the pricing rule λ_t also ensures that $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$. The problem is that γ_t becomes negative as $t \to \infty$, which is a contradiction because the insider can always guarantee herself nonnegative continuation payoffs even when $P_t = V_t$. That is, setting $T = \infty$ implicitly involves making undeliverable promises (negative continuation values) to the insider. Thus, a switching time T must exist. T is the time it takes for the variance to decrease from Σ_0 to $\widetilde{\Sigma}$.

$\widetilde{\Sigma}$ separates the indifference states from the corner states. We already argued above that the critical state $\widetilde{\Sigma}$ cannot be increased. We now argue that it cannot be decreased either. Let $\hat{\Sigma} \in (0, \widetilde{\Sigma})$ and suppose we make $[0, \hat{\Sigma}]$ the corner states and $(\hat{\Sigma}, \infty)$ the indifference states. Assume $\hat{\Sigma} < \Sigma_0 < \widetilde{\Sigma}$. Then, in equilibrium, the insider would be kept indifference until Σ_t decreases to $\hat{\Sigma}$ and then she would trade at maximal rate afterwards. But, since $2\lambda(\hat{\Sigma})\alpha(\hat{\Sigma}) < 1$ and $\lambda(\Sigma_t)$ and $\alpha(\Sigma_t)$ must be continuous, $2\lambda(\Sigma_t)\alpha(\Sigma_t) < 1$ for Σ_t slightly larger than $\hat{\Sigma}$, which is a contradiction because the insider would not be indifferent at Σ_t, as required by the equilibrium.

The informal argument above establishes that T increases with Σ_0; the larger is Σ_0, the more time it takes to reduce the variance to $\widetilde{\Sigma}$. On the other hand, $\widetilde{\Sigma}$ increases with σ_y.[9] Therefore, keeping Σ_0 constant, if we increase σ_y, T decreases. We study a similar effect for σ_v in Proposition 3 below.

4 Limit Equilibrium

We now relax the arbitrary constraint on the insider's trading rate and let $\bar{\beta}$ go to infinity to construct a limit equilibrium. This limit equilibrium *is not* an equilibrium of the unconstrained continuous-time model. However, we will argue in Section 5 that the limit equilibrium approximates the equilibrium of any discrete-time model with period length sufficiently short.

According to Theorem 1, as $\bar{\beta}$ grows large so does the insider trading rate in $[T, \infty)$. In the

---

[9] In the proof of Theorem 1 in the Appendix we show that $h(k)$ is increasing at $\tilde{k}$, when $h(\tilde{k}) = 1$. If σ_y increases, L decreases and $h(x)$ decreases for each x. Therefore, $\tilde{k}$ and $\widetilde{\Sigma}$ increase.

limit as $\bar{\beta} \to \infty$, the insider wants to trade at an infinite rate exerting control of unbounded variation over the price dynamics. Nevertheless, the following Theorem guarantees that both $\Pi(t, M, \bar{\beta})$ and $M_t(\bar{\beta})$ admit a well-defined limit as $\bar{\beta} \to \infty$.

Theorem 2 *For each $\bar{\beta} > 0$, let $(k(\bar{\beta}), \lambda_0(\bar{\beta}), \gamma_0(\bar{\beta}), T(\bar{\beta}))$ be the equilibrium associated with $\bar{\beta}$ specified in Theorem 1. Then $(k(\bar{\beta}), \lambda_0(\bar{\beta}), \gamma_0(\bar{\beta}), T(\bar{\beta})) \to (0, \lambda_0, \gamma_0, T)$ as $\bar{\beta} \to \infty$, where $\lambda_0 = \frac{\sigma_v}{\sigma_y}\, e^{\mu T}$ and T is the unique nonnegative root of the equation*

$$\Sigma_0 + \sigma_v^2\, T = \sigma_v^2 \left[\frac{e^{2\,\mu\, T} - 1}{2\,\mu}\right].$$

The limit equilibrium associated with $(0, \lambda_0, \gamma_0, T)$ has two phases separated by the switching time T.

- Absolutely Continuous Phase in $[0, T)$: *In this phase, the insider's trading strategy and market maker's pricing rule are given by*

$$\mathrm{d}X_t = \beta_t\, (V_t - P_t)\, \mathrm{d}t \qquad \textit{and} \qquad \mathrm{d}P_t = \lambda_t\, \mathrm{d}Z_t \qquad t < T,$$

where β_t and λ_t are the two deterministic functions

$$\beta_t = \frac{\sigma_v\, \sigma_y\, e^{\mu(T-t)}}{\Sigma_t} \qquad \textit{and} \qquad \lambda_t = \frac{\sigma_v}{\sigma_y}\, e^{\mu\,(T-t)}, \quad t < T.$$

The variance of the market maker's estimate of V_t is given by

$$\Sigma_t = \Sigma_0 + \sigma_v^2\, t - \sigma_v^2 e^{2\mu T} \left[\frac{1 - e^{-2\,\mu\, t}}{2\mu}\right] \qquad t < T,$$

which decreases monotonically to 0 in $[0, T)$.

- Unbounded Variation Phase in $[T, \infty)$: *In this phase, the (inverse) market depth is constant: $\lambda_t = \sigma_v/\sigma_y$, and the market maker's pricing rule satisfies*

$$\mathrm{d}P_t = \frac{\sigma_v}{\sigma_y}\, \mathrm{d}Z_t, \qquad t \geq T.$$

Moreover, the price differential $M_t(\bar{\beta})$ converges weakly to 0 over compacts in $[T, \infty)$. As a result, the insider's trading strategy converges weakly to $X_t = X_T + \sigma_y\, [(B_t^v - B_T^v) - (B_t^y - B_T^y)]$ and $\Sigma_t = 0$ for all $t \geq T$.

- Insider's Payoff: *Let* $(T-t)^+ = \max\{0, T-t\}$*. As* $\bar\beta \to \infty$*, the insider's value function* $\Pi(t, M_t, \bar\beta)$ *converges to the quadratic function*

$$\Pi(t, M_t) = \alpha_t\, M_t^2 + \gamma_t \qquad \textit{where} \qquad \alpha_t = \frac{\sigma_y}{2\,\sigma_v}\, e^{-\mu\,(T-t)^+} \qquad \textit{and}$$

$$\gamma_t = \frac{\sigma_y\,\sigma_v}{4\,\mu}\left[3\,e^{-\mu\,(T-t)^+} + e^{\mu\,(T-t)^+}\right] + \frac{\sigma_y\,\sigma_v}{2}\,(T-t)^+\,e^{-\mu\,(T-t)^+}.$$

In particular, in the limit equilibrium, for any $t \geq T$*, the insider's expected continuation value is* $\Pi(t, 0) = \gamma_T = \sigma_y\,\sigma_v/\mu$*.*

The previous result summarizes a number of important features of the limit equilibrium. One of the remarkable properties of this equilibrium is the existence a finite time T, endogenously determined, such that $M_t \Rightarrow 0$ for $t \geq T$. Thus, the market maker fulfills his obligation in a rather strong sense after T. He is concerned with setting prices so that $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$. Theorem 2 implies that P_t converges uniformly on compact sets to V_t in $[T, \infty)$.[10] Hence, in the limit as $\bar\beta \to \infty$, the market maker knows exactly the intrinsic value of the asset and the price reflects this value at all times $t \geq T$. Despite this market efficiency, the insider collects positive rents ($\Pi(t, 0) = \gamma_T$) in $[T, \infty)$ because she can continuously observe V_t and trade unbounded amounts each time V_t separates itself (marginally) from P_t. As a result, after T, the insider trading volume X_t behaves as a Brownian motion and has unbounded variation. It is also interesting to note that $X_t - X_T$ is independent of σ_v.

When the inflow of new information is small (for example, when $\sigma_v = 0$ because V_t is constant or the insider cannot track V_t after $t = 0$), the insider would collect small rents after the market reaches full efficiency. Therefore, the insider spends instead her private information slowly and market efficiency is reached only asymptotically ($T = \infty$). This and other properties of the equilibrium are summarized in the following theorem. The theorem also establishes that in this case the limit equilibrium in Theorem 2 is effectively an equilibrium of the continuous-time game if we restrict the insider's strategy to the space $\mathcal{B}$ of trade rates β such that

$$\mathbb{E}\left[\int_0^\infty e^{-\mu\,t}\,|\beta_t|\,M_t^2\,\mathrm{d}t\right] < \infty. \tag{11}$$

Condition (11) rules out some bluffing schemes where the insider trades in the "wrong" direction and accumulates unbounded losses.

---

[10] This follows from the Skorohod Representation Theorem and the fact that $M_t = V_t - P_t$ converges weakly to (the continuous process) 0.

Theorem 3 *Suppose the asset's volatility $\sigma_v(t)$ is a function of time, and let Γ_t be the insider's cumulative inflow of private information from time t onwards, that is,*

$$\Gamma_t = \int_t^\infty \sigma_v^2(t)\,\mathrm{d}t.$$

Assume that $\Gamma_0 < \infty$ and $(\Sigma_0 + \Gamma_0)\, e^{-2\,\mu\, t} > \Gamma_t$ for all t. When the insider's strategy space is constrained by (11), there exists a continuous-time equilibrium that satisfies

$$\Sigma_t = (\Sigma_0 + \Gamma_0)\, e^{-2\,\mu\, t} - \Gamma_t, \qquad \lambda_t = \sqrt{\frac{2\,\mu\,(\Sigma_0 + \Gamma_0)}{\sigma_y^2}}\; e^{-\mu\, t}, \qquad \beta_t = \frac{\sigma_y^2\,\lambda_t}{\Sigma_t}, \tag{12}$$

$$\alpha_t = \frac{e^{\mu\, t}}{2}\sqrt{\frac{\sigma_y^2}{2\,\mu\,(\Sigma_0 + \Gamma_0)}} \qquad \textit{and} \qquad \gamma_t = \alpha_t\,\Gamma_t + \frac{\sigma_y^2\,\lambda_t}{4\,\mu}, \tag{13}$$

Under the conditions of Theorem 3, in equilibrium $\Sigma_t \downarrow 0$ as $t \to \infty$, but $\Sigma_t > 0$ for all $t \geq 0$. More importantly, the trading rate β_t remains bounded for all $t \geq 0$ so the insider's strategy is a process of bounded variation. If the insider's strategy space is not constrained by (11), then (12) and (13) still define the limit equilibrium, but this may fail to be an equilibrium. Theorem 3 again highlights the role of the flow of information. When this flow is substantial, the insider is happy to trade quickly to exploit current arbitrage opportunities. Even though in the process she "informs" the market about what she knows now, new arbitrage opportunities will develop soon. In the limit equilibrium, she transfers all her information (initial + flow) by time T. However, when this flow is relatively low, she is not willing to trade that fast.

In an equilibrium of the constrained continuous-time model, the market maker's expected payoff is 0. This property is preserved in the limit equilibrium. Thus, the liquidity traders' expected loss must equal the insider's expected profit. This amount is computed in the following proposition.

Proposition 2 *In the limit equilibrium, the insider's ex-ante (at time 0, before observing any signals) expected payoff-to-go is*

$$\mathbb{E}[\Pi(t, M_t)] = \frac{\sigma_v\,\sigma_y}{\mu}\,\cosh(\mu(T - t)^+).$$

$\mathbb{E}[\Pi(t, M_t)]$ is the market's best estimate of the insider's expected continuation payoff from time t on. The insider's expected payoff decreases monotonically with time in $[0, T)$ and stays constant after T. Thus, liquidity traders that place their orders late in the game expect to make smaller losses.

One can show that the limit equilibrium satisfies the smooth-pasting condition

$$\lim_{t \uparrow T} \dot{\Sigma}_t = 0.$$

This is in contrast to the equilibria obtained in models that assume a fixed announcement date (*e.g.*, Kyle (1985)), where Σ_t does not approach 0 smoothly.

To get a sense of how likely is that market efficiency is reached in the limit equilibrium, let us compare T and the average time at which the announcement date occurs, $1/\theta$. From the definition of T in Theorem 2, we can show that

$$T \leq \frac{1}{\theta} \qquad \text{if} \qquad \Sigma_0 \leq \left(\frac{e^2 - 3}{2} \right) \frac{\sigma_v^2}{\theta} \sim 2 \, \frac{\sigma_v^2}{\theta}.$$

(The inequalities are tight if the discount factor δ is 0.) Roughly speaking, the previous inequalities suggest that on average market efficiency is reached when the insider's initial (lumpy) private information Σ_0 is less than twice her average cumulative inflow of new private information σ_v^2/θ. Furthermore, one can show that as $\sigma_v \to \infty$ the switching time T converges to 0 and market efficiency is reach instantaneously. On the other hand, as $\sigma_v \downarrow 0$, the switching time T diverges to $+\infty$, efficiency is never reached and the resulting profile coincides with the equilibrium derived in Theorem 3. The volatility coefficient σ_v determines the amount of information asymmetry between the insider and the rest of the market. The higher is σ_v, the faster the insider reveals her information, but also the larger is her profit.

Proposition 3 *The value of* Σ_t *weakly decreases with* σ_v *for all* $t \geq 0$. *On the other hand, the insider's ex-ante expected payoff* $\mathbb{E}[\Pi(t, M_t)] = \alpha_t \, \Sigma_t + \gamma_t$ *is weakly increasing in* σ_v *for all* $t \geq 0$.

The more volatile is the fundamental value, the faster the price adjusts to the current intrinsic value. However, this efficiency come at a cost. Indeed, the insider is willing to trade away her private information faster because the market maker compensates her for doing so. Hence, we expect market prices to be more informative when the volatility of the fundamental value is higher. For example, in the special case in which there is no volatility ($\sigma_v = 0$), market efficiency ($\Sigma_t = 0$) is reached only asymptotically as $t \to \infty$ and the insider's ex-ante payoff is minimized.

4.1 Market Efficiency and Equilibrium

In what follows, we show that in general, the limit equilibrium cannot be an equilibrium of the unconstrained continuous time model.

Recall that the insider's expected payoff-to-go after time T can be written as

$$\Pi(T, M_T, P, X) = \mathbb{E}\left[\int_T^\infty e^{-\mu\,(t-T)} M_t\,\mathrm{d}X_t + \int_T^\infty e^{-\mu\,(t-T)}\,\mathrm{d}[X,V]_t - \int_T^\infty e^{-\mu\,(t-T)}\,\mathrm{d}[X,P]_t\right].$$

Let $(P^{\bar\beta}, X^{\bar\beta})$ be the equilibrium constructed in Theorem 1 and (P, X) be the limit equilibrium derived in Theorem 2. After time T, the market maker's pricing strategy P is given by $\mathrm{d}P_t = \lambda_T \mathrm{d}Z_t$, where $\lambda_T = \sigma_v/\sigma_y$, and the insider's cumulative volume of trade is a martingale process such that $\mathrm{d}X_t = \sigma_y\,[\mathrm{d}\,B^v_t - \mathrm{d}B^y_t]$. Thus, when $M_T = 0$, the first stochastic integral with respect to X_t has 0 expectation and the quadratic covariations between X_t and V_t and between X_t and P_t satisfy $\mathrm{d}[X,V]_t = \sigma_y\,\sigma_v\,\mathrm{d}t$ and $\mathrm{d}[X,P]_t = \lambda_t\,\sigma_y^2\mathrm{d}t = \sigma_y\,\sigma_v\,\mathrm{d}t$ respectively. It follows that

$$\Pi(T, 0, P, X) = 0 \neq \lim_{\bar\beta\to\infty} \Pi(T, 0, P^{\bar\beta}, X^{\bar\beta}) = \frac{\sigma_v\sigma_y}{\mu}.$$

That is, Π has a discontinuity at (P, X) as X is approached by strategies of bounded variation. For any $\bar\beta > 0$, the equilibrium of the constrained model is such that the insider makes positive rents after the switch time T, when she trades at the maximal rate. As $\bar\beta \uparrow \infty$, those rents converge to $\sigma_v\,\sigma_u/\mu > 0$. However, in the limit equilibrium, the insider makes no profits after T.

Since $\Pi(T, 0, P, X) = 0$, (P, X) is not an equilibrium because, given the pricing strategy P, the insider clearly has trading strategies with strictly positive expected payoff. Thus X is not a best reply to P.

5 Discrete-Time Model

We now discuss the relationship between the limit equilibrium and the equilibrium of a game in which the insider trades at discrete time epochs. Specifically, we show that the limit equilibrium is indeed a good approximation for the discrete-time equilibrium as the period length goes to 0. In what follows we provide a brief description of the discrete-time model and its equilibrium. For a detailed discussion of this model we refer the reader to Caldentey and Stacchetti (2007).

In the discrete-time model, the market maker opens the floor for trading only at discrete times $\{t_n\}_{n\geq 0}$. These trading dates are evenly spaced over time (*e.g.*, once a day) so that $t_n = n\,\Delta$ for some positive constant Δ. The interval of time $[t_n, t_{n+1})$ is called period n. We let x_n and y_n be the orders placed simultaneously by the insider and liquidity traders, respectively, at trading time t_n. We also denote their cumulative trading up to time t_n by $X_n = \sum_{k=0}^n x_k$ and $Y_n = \sum_{k=0}^n y_k$, respectively. As before, we assume that the market maker is not able to differentiate between insider and liquidity trading. He observes only the net volume of trade $z_n = x_n + y_n$ and the net market holding $Z_n = X_n + Y_n$ at every trading time t_n.

Consistent with the continuous-time setting, we assume that the insider observes the evolution of the asset value. That is, at every trading epoch t_n, and before placing her order, she observes the intrinsic value V_n which evolves as a random walk with i.i.d. increments that are normally distributed with variance $\Sigma_v = \sigma_v^2 \, \Delta$. Similarly, we assume that liquidity trading follows a random walk: in particular $\{y_n\}$ is a sequence of i.i.d. normal random variables with mean 0 and variances $\Sigma_y = \sigma_y^2 \, \Delta$.

At the beginning of each period n, before the fundamental value becomes public knowledge, the market maker commits to a pricing rule (that is legally binding). The rule specifies the price P_n for the current period's transactions as a function of the total volume of trade z_n. The insider and the liquidity traders place their orders after the rule is announced. All orders are executed at the end of the period. Note that while the market maker commits to a rule before knowing the current period's volume of trade, the actual price is determined after learning the volume of trade.

Given a strategy $\{X_n\}$ for the insider's holding and market prices $\{P_n\}$, the insider's expected payoff is

$$\Pi(P, X) = \mathbb{E}\Big[\sum_{n=0}^{\nu} e^{-n\,\delta\Delta} \, (V_n - P_n) \, x_n\Big],$$

where ν denotes the period when the value of the asset becomes public knowledge. (We assume that this announcement always occurs at the end of the period, after the current orders are executed.) The discrete random variable ν has a geometric distribution with probability of failure $q = e^{-\theta\Delta}$.

We will restrict attention to *linear Markovian* equilibria with a particular state space. At the beginning of period n, before the market maker observes the volume of trade, the state is $(n, v_{n-1}, \Sigma_{n-1})$, where v_{n-1} is the market maker's estimate of V_n and Σ_{n-1} is the variance of this estimate. Since the market maker's estimate of V_n depends on the strategy X of the insider, the state and corresponding Markovian strategy profile need to be specified simultaneously.

The following definition formalizes the notion of a linear Markovian profile that we will use to characterize the equilibria of this discrete-time model.

Definition 3 (Linear Markovian Equilibrium) *We say that a strategy (P, X) is a linear Markovian profile if there exist two sequences of functions $\{\lambda_n\}$ and $\{\beta_n\}$ such that the price*

P_n and the insider trading x_n, in period n, satisfy

$$P_n = v_{n-1} + \lambda_n(\Sigma_{n-1}) z_n \quad \textit{and} \quad x_n = \beta_n(\Sigma_{n-1})(V_n - v_{n-1}). \tag{14}$$

In addition, the profile (P, X) is an equilibrium if (i) *$P_n = \mathbb{E}[V_n | X, z_k, k = 0, 1, \ldots, n]$ for any $n \geq 0$ and* (ii) *given P, $\Pi(P, X)$ is bounded above and $\{X_n\}$ maximizes $\Pi(P, X)$.*

With the notation above, we can express $v_n = \mathbb{E}[V_{n+1} | X, z_k, k = 0, 1, \ldots, n]$ and $\Sigma_n = \mathbb{E}[(V_{n+1} - v_n)^2 | X, z_k, k = 0, 1, \ldots, n]$. Since $V_{n+1} - V_n$ is an independent normal random variable, $P_n = v_n$. Before the game starts, v_{-1} is the market maker's estimate of V_0 and Σ_{-1} is the variance of this estimate.

We characterize a unique linear Markovian equilibrium in the following theorem, whose statement requires some additional notation and whose proof can be found in Caldentey and Stacchetti (2007). Let $\Pi_n(p, \Sigma, V)$ be the insider's expected payoff-to-go from period n onwards conditional on $V_n = V$, $P_{n-1} = p$ and $\Sigma_{n-1} = \Sigma$. We also define $\rho = e^{-\mu\Delta}$ with $\mu = \theta + \delta$.

Theorem 4 *There exist unique sequences $\{\lambda_n\}$, $\{\beta_n\} \in \mathbb{R}_{++}$ such that the linear strategy profile (P, X) defined by (14) is a Markovian equilibrium. In equilibrium, $\{\Sigma_n\}$ is a deterministic trajectory that is not affected by the (stochastic) choices of the insider and the market maker. Starting from (Σ_{-1}, β_0), the values of λ_n, β_n and Σ_n are determined recursively solving*

$$\Sigma_n = \Sigma_v + \frac{\Sigma_{n-1}\Sigma_y}{\beta_n^2 \Sigma_{n-1} + \Sigma_y}, \quad \lambda_n = \frac{\beta_n \Sigma_{n-1}}{\beta_n^2 \Sigma_{n-1} + \Sigma_y}, \quad \beta_{n+1}\Sigma_n = \rho \beta_n \Sigma_{n-1} \left[\frac{\Sigma_y^2}{\Sigma_y^2 - \beta_n^4 \Sigma_{n-1}^2} \right]. \tag{15}$$

Furthermore, there exist sequences $\{\alpha_n\}, \{\gamma_n\} \subset \mathbb{R}_{++}$ such that the insider's expected payoff-to-go Π_n satisfies

$$\rho\, \Pi_n(p, \Sigma, V) = \alpha_n \left(V - p\right)^2 + \gamma_n \qquad \textit{for all } n \geq 0, \tag{16}$$

where

$$\frac{\alpha_n}{\rho} = [4\lambda_n(1 - \lambda_n \alpha_{n+1})]^{-1} \quad \textit{and} \quad \frac{\gamma_n}{\rho} = \gamma_{n+1} + \alpha_{n+1}(\Sigma_v + \lambda_n^2 \Sigma_y). \tag{17}$$

We devote the remainder of this section to showing that the discrete-time linear Markovian equilibrium in discrete-time converges to the limit equilibrium of Theorem 2 as $\Delta \downarrow 0$.

Given $\Delta > 0$, let us denote by $\{(\Sigma^{\Delta}_{n-1}, \beta^{\Delta}_{n})\}_{n\geq 0}$ the corresponding discrete-time profile generated by the recursions in (15) with initial conditions $(\Sigma^{\Delta}_{-1}, \beta^{\Delta}_{0})$. In the analyis that follows, the value of Σ^{Δ}_{-1} is always set equal to the same initial value Σ_0, independent of Δ. We can extend $\{(\Sigma^{\Delta}_{n-1}, \beta^{\Delta}_{n})\}_{n\geq 0}$ to a continuous-time profile $\{(\Sigma^{\Delta}_{t}, \beta^{\Delta}_{t})\}_{t\geq 0}$ for all calendar time $t \geq 0$ defined (with a small abuse of notation) as follows: $\Sigma^{\Delta}_{t} = \Sigma^{\Delta}_{\lfloor t-\Delta \rfloor}$ and $\beta^{\Delta}_{t} = \beta^{\Delta}_{\lfloor t \rfloor}/\Delta$, where $\lfloor t \rfloor$ denotes the largest integer such that $n\Delta \leq t$.

The profile $(\Sigma^{\Delta}_{t}, \beta^{\Delta}_{t})$ is consistent with the equilibrium conditions if it is nonnegative and satisfies the recursion in (15), that is,

$$\Sigma^{\Delta}_{t+\Delta} = \sigma_v^2 \Delta + \frac{\sigma_y^2\, \Sigma^{\Delta}_{t}}{(\beta^{\Delta}_{t})^2 \Sigma^{\Delta}_{t}\, \Delta + \sigma_y^2} \quad \text{and} \quad \beta^{\Delta}_{t+\Delta} \Sigma^{\Delta}_{t+\Delta} = \frac{e^{-\mu\,\Delta}\, \beta^{\Delta}_{t}\, \sigma_y^4\, \Sigma^{\Delta}_{t}}{\sigma_y^4 - (\beta^{\Delta}_{t})^4\, (\Sigma^{\Delta}_{t})^2\, \Delta^2}\,, \qquad t \geq 0. \tag{18}$$

Given Σ_0, the profile $(\Sigma^{\Delta}_{t}, \beta^{\Delta}_{t})$ is uniquely specified by β^{Δ}_{0} and (18). The pair of difference equations in (18) can be rewritten as

$$\frac{\Sigma^{\Delta}_{t+\Delta} - \Sigma^{\Delta}_{t}}{\Delta} = \sigma_v^2 - \frac{(\beta^{\Delta}_{t}\, \Sigma^{\Delta}_{t})^2}{\sigma_y^2 + (\beta^{\Delta}_{t})^2\, \Sigma^{\Delta}_{t}\, \Delta} \qquad \text{and} \tag{19}$$

$$\frac{\beta^{\Delta}_{t+\Delta}\, \Sigma^{\Delta}_{t+\Delta} - \beta^{\Delta}_{t}\, \Sigma^{\Delta}_{t}}{\Delta} = \left[\frac{\sigma_y^4\, (e^{-\mu\,\Delta} - 1)/\Delta + (\beta^{\Delta}_{t})^4\, (\Sigma^{\Delta}_{t})^2\, \Delta}{\sigma_y^4 - (\beta^{\Delta}_{t})^4\, (\Sigma^{\Delta}_{t})^2\, \Delta^2} \right] \beta^{\Delta}_{t}\, \Sigma^{\Delta}_{t}\,, \tag{20}$$

which suggests that these equations can be approximated by a pair of differential equations for Δ sufficiently small. For a given t, suppose that $\limsup_{\Delta\downarrow 0}\, (\beta^{\Delta}_{t})^2\, \Sigma^{\Delta}_{t} < \infty$. Then, $(\beta^{\Delta}_{t})^2\, \Sigma^{\Delta}_{t}\, \Delta$ is negligible for Δ sufficiently small and, as $\Delta \downarrow 0$, (19)-(20) converge to

$$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \sigma_v^2 - \frac{(\Sigma_t\, \beta_t)^2}{\sigma_y^2} \qquad \text{and} \qquad \frac{\mathrm{d}(\Sigma_t\, \beta_t)}{\mathrm{d}t} = -\mu\, \Sigma_t\, \beta_t. \tag{21}$$

Since the condition $\limsup_{\Delta\downarrow 0}\, (\beta^{\Delta}_{t})^2\, \Sigma^{\Delta}_{t} < \infty$ is trivially satisfied at $t = 0$, we expect (by continuity) that the convergence above holds for all $t \in [0, \tau)$ for some positive $\tau > 0$. Then, integrating (21) in this range we obtain a continuous profile (Σ^0_t, β^0_t) given by

$$\Sigma^0_t = \Sigma_0 + \sigma_v^2\, t - \frac{(\beta_0\, \Sigma_0)^2}{2\,\mu\, \sigma_y^2}\, (1 - e^{-2\,\mu\, t}) \quad \text{and} \quad \beta^0_t = \frac{\beta_0\, \Sigma_0\, e^{-\mu\, t}}{\Sigma^0_t}, \qquad t < \tau. \tag{22}$$

Note that for this continuous-time solution, the condition $\limsup_{\Delta\downarrow 0}\, (\beta^{\Delta}_{t})^2\, \Sigma^{\Delta}_{t} = (\beta^0_t)^2\, \Sigma^0_t < \infty$ reduces to $\Sigma^0_t > 0$. Hence, τ is uniquely determined as the smallest (positive) solution of the equation $\Sigma^0_\tau = 0$, that is,

$$0 = \Sigma_0 + \sigma_v^2\, \tau - \frac{(\beta_0\, \Sigma_0)^2}{2\,\mu\, \sigma_y^2}\, (1 - e^{-2\,\mu\,\tau}).$$

We will denote this solution by $\tau(\beta_0)$ to emphasize its dependence on the initial condition β_0. If such a solution does not exist then we set $\tau(\beta_0) = \infty$ and the continuous-time approximation holds for all $t \geq 0$.

According to the previous discussion, the discrete-time equilibrium $(\Sigma_t^\Delta, \beta_t^\Delta)$ converges to (Σ_t^0, β_t^0) for all $t \in [0, \tau(\beta_0))$. However, this analysis, based on the limiting behavior of the difference equations (19)-(20), is not sufficient to pinpoint the right value of β_0 to initialize the continuous-time solution (Σ_t^0, β_t^0). For this, we need to impose the conditions that characterize the unique discrete-time equilibrium of Theorem 4.

The initial value β_0^Δ for the discrete-time equilibrium with period length Δ can be found as the maximum of the set $\mathcal{B}^\Delta(\Sigma_0)$. This is the set of all those β_0^Δ for which the corresponding profile $(\Sigma_t^\Delta, \beta_t^\Delta)$ generated by (18) is nonnegative for all t. While every $\beta_0^\Delta \in \mathcal{B}^\Delta(\Sigma_0)$ generates a feasible trajectory $(\Sigma_t^\Delta, \beta_t^\Delta)$, only $\max\{\mathcal{B}^\Delta(\Sigma_0)\}$ ensures that $\Pi(P, X) < \infty$. (For more details see Lemma 3 in Caldentey and Stacchetti 2007). Hence, we need to characterize the limit of the set $\mathcal{B}^\Delta(\Sigma_0)$ as $\Delta \downarrow 0$. Let us denote this limiting set by $\mathcal{B}^0(\Sigma_0) = \liminf_{\Delta \downarrow 0} \mathcal{B}^\Delta(\Sigma_0)$. That is, $\beta_0 \in \mathcal{B}^0(\Sigma_0)$ if and only if there exists $\hat{\Delta} > 0$ such that $\beta_0 \in \mathcal{B}^\Delta(\Sigma_0)$ for all $\Delta \leq \hat{\Delta}$.

Proposition 4 *Let $\eta \geq 2$ be the unique root of the equation*

$$0 = \Sigma_0 + \frac{\sigma_v^2}{2\,\mu}\,[2 + \ln(\eta - 1) - \eta].$$

Then $\mathcal{B}^0(\Sigma_0) = [0, \bar{\beta}_0)$ where $\bar{\beta}_0 = \frac{\sigma_v\,\sigma_y}{\Sigma_0}\sqrt{\eta - 1}$. Furthermore, $\tau(\bar{\beta}_0) = \frac{1}{\mu}\ln\left(\frac{\Sigma_0\,\bar{\beta}_0}{\sigma_v\,\sigma_y}\right)$.

The proof of the proposition can be found in Caldentey and Stacchetti (2007).

According to this result, as $\Delta \downarrow 0$, the initial condition β_0^Δ that characterized the discrete-time equilibrium converges to $\bar{\beta}_0$. It follows then from our previous discussion that for all $t < \tau(\bar{\beta}_0)$,

$$\lim_{\Delta \downarrow 0}\ \Sigma_t^\Delta = \Sigma_t^0 = \Sigma_0 + \sigma_v^2\,t - \frac{(\Sigma_0\,\bar{\beta}_0)^2}{2\,\mu\,\sigma_y^2}(1 - e^{-2\,\mu\,t}) \quad \text{and} \quad \lim_{\Delta \downarrow 0}\ \beta_t^\Delta = \beta_t^0 = \frac{\Sigma_0\,\bar{\beta}_0}{\Sigma_t^0}\,e^{-\mu\,t}.$$

The proof of Proposition 4 reveals that at $t = \tau(\bar{\beta}_0)$, both $\Sigma_t^0 = 0$ and $\dot{\Sigma}_t^0 = 0$. That is, the limit equilibrium satisfies a smooth pasting condition.

One can also show that $\tau(\bar{\beta}_0)$ is equal to T in Theorem 2 and that the limiting solution (Σ_t^0, β_t^0) above coincides with the limit equilibrium (Σ_t, β_t) derived in Theorem 2 for $t < T$. In other words, for Δ small, the discrete-time equilibrium is well approximated by the *absolutely continuous phase* of the limit equilibrium.

As for what happens after time T, we note that there is no real discrete-time counterpart for the *unbounded variation phase*. In fact, for the continuous-time model, T is the time at which the system reaches stationarity. In discrete time, however, stationarity is reached only asymptotically (unless $\Sigma_0 = 0$). Nevertheless, the continuous-time variance process Σ_t in Theorem 2 is a good approximation of the discrete-time variance Σ_t^Δ for all $t \geq 0$. Indeed, for Δ sufficiently small, the discrete-time variance Σ_t^Δ is nonincreasing in t and, since $\Sigma_t = 0$ for all $t \geq T$, it follows that $|\Sigma_t^\Delta - \Sigma_t| \leq |\Sigma_T^\Delta - \Sigma_T|$ for all $t \geq T$. Since $\Sigma_T^\Delta \to \Sigma_T = 0$ as $\Delta \downarrow 0$ we conclude that

$$\lim_{\Delta \downarrow 0} \sup_{t \geq 0} |\Sigma_t^\Delta - \Sigma_t| = 0.$$

In summary, the limiting equilibrium of Theorem 2 is a good approximation of the equilibrium of the discrete-time model when Δ is small.

6 Conclusions

The paper introduces a model that combines a random announcement time with an insider who receives a flow of information as well as an initial signal. The new model produces a (limit) equilibrium with novel features. The insider is confronted with two distinct incentive regimes. Before the endogenous time T, she is indifferent about how to consume her information stock, which includes the initial signal and the flow information she receives in the interval $[0, T]$. Nevertheless, in equilibrium she exhausts all this stock by time T, so that the market reaches full efficiency at time T. After T she is eager to exhaust any additional piece of information immediately. As she does, she keeps the market fully informed until the announcement date, which reveals no further information.

The flow of new information, that in principle exacerbates the informational asymmetry, in equilibrium induces the insider to release her information faster. As an intriguing consequence, the market is uniformly better informed and reaches full efficiency earlier when this source of informational asymmetry (the variation of the innovation process) is larger.

The analysis also exposes a potential vulnerability of continuous-time models. The natural discrete-time model has an equilibrium that, albeit difficult to construct explicitly, has a well defined limit as the period length decreases to 0. However, this limit equilibrium *is not* an equilibrium of the corresponding continuous-time model. The same limit equilibrium is recovered as the limit of equilibria of a sequence of constrained continuous-time models, where the insider's rate of transaction is exogenously bounded.

Appendix

Proof of Proposition 1.

The condition $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$ implies that P_t is the orthogonal projection V_t on $\mathcal{F}_t^M$ in L^2. Hence, we can interpret the market maker's equilibrium condition as the solution to a classical Kalman-Bucy filtering problem. Let the signal process be the value of the fundamental V_t, with dynamics

$$\mathrm{d}V_t = \sigma_v \, \mathrm{d}B_t^v,$$

and the observation process be the price process P_t, with dynamics

$$\mathrm{d}P_t = \lambda_t \, \mathrm{d}Z_t = \beta_t \, \lambda_t \, (V_t - P_t) \, \mathrm{d}t + \sigma_y \, \lambda_t \, \mathrm{d}B_t^y.$$

Let v_t be the corresponding optimal (in mean square sense) filtering estimate of V_t and Σ_t be the filtering error. Then, the equilibrium condition is $P_t = v_t$.

The generalized Kalman filter conditions for the pair (V_t, P_t) are given by

$$\mathrm{d}v_t = \frac{\Sigma_t \, \beta_t}{\lambda_t \, \sigma_y^2} \, [\mathrm{d}P_t - \lambda_t \, \beta_t \, (v_t - P_t) \, \mathrm{d}t] \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \frac{(\Sigma_t \, \beta_t)^2}{\sigma_y^2}.$$

To recover the identity $P_t = v_t$ we need to impose that

$$\frac{\Sigma_t \, \beta_t}{\lambda_t \, \sigma_y^2} = 1 \qquad \text{or equivalently} \qquad \Sigma_t \, \beta_t = \lambda_t \, \sigma_y^2.$$

This equality together with the border condition $v_0 = P_0$ imply that $v_t = P_t$ for all $t > 0$. This equality also implies that $(\Sigma_t \, \beta_t)^2 = \lambda_t^2 \, \sigma_y^4$. Therefore, the second filtering condition leads to the differential equation

$$\dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2 \, \lambda_t^2,$$

which completes the proof of the Lemma. ▮

Proof of Theorem 1.

To prove this theorem we establish that (i) λ_t satisfies the filtering condition (5) for the market-maker; and (ii) given λ_t, $(\beta_t, \Pi(t, M))$ solves the HJB equation (4).

Filtering Conditons: In $[0, T)$, (6) and (7) imply that $\Sigma_t \, \beta_t = \sigma_y^2 \, \lambda_t = \sigma_y^2 \, \lambda_0 \, e^{-\mu \, t}$, and the second filtering condition of (5) is satisfied if and only if

$$\dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2 \, \lambda_t^2 = \sigma_v^2 - \sigma_y^2 \, \lambda_0^2 \, e^{-2 \, \mu \, t}.$$

The solution of this differential equation is

$$\Sigma_0 + \sigma_v^2 t - \frac{\lambda_0^2\,\sigma_y^2}{2\mu}\big[1 - e^{-2\,\mu\,t}\big].$$

In (T, ∞), $\beta_t \equiv \bar{\beta}$. In this case, the filtering conditions (5) become

$$\Sigma_t\,\bar{\beta} = \lambda_t\,\sigma_y^2 \qquad \text{and} \qquad \dot{\Sigma}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2.$$

The second filtering equation leads to the differential equation

$$\frac{\sigma_y^2}{\bar{\beta}}\dot{\lambda}_t = \sigma_v^2 - \sigma_y^2\,\lambda_t^2.$$

Therefore, the filtering conditions are satisfied if and only if

$$\lambda_t = \frac{\sigma_v}{\sigma_y}\left[\frac{e^{L\,t} + k\,e^{L\,T}}{e^{L\,t} - k\,e^{L\,T}}\right] \qquad \text{and} \qquad \Sigma_t = \frac{\sigma_v\,\sigma_y}{\bar{\beta}}\left[\frac{e^{L\,t} + k\,e^{L\,T}}{e^{L\,t} - k\,e^{L\,T}}\right],$$

for some constant of integration k. Note that λ_t and Σ_t are decreasing in t if and only if $k \geq 0$, a fact we establish below.

OPTIMALITY CONDITONS: We prove only Case I. The proof of Case II follows directly from this derivation.

We guess that $\Pi(t, M) = \alpha_t M^2 + \gamma_t$, with the coefficients α_t and γ_t defined by (8) and (9). Then, the HJB equation becomes

$$0 = \max_{|\beta|\leq\bar{\beta}}\left\{\left[\beta\,(1 - 2\lambda_t\,\alpha_t) + \dot{\alpha}_t - \mu\,\alpha_t\right]M^2 + \alpha_t(\sigma_v^2 + \lambda_t^2\sigma_y^2) + \dot{\gamma}_t - \mu\gamma_t\right\}.$$

Note that the right-hand side is linear in β, and recall that $\sigma_t^2 = \sigma_v^2 + \lambda_t^2\sigma_y^2$.

In $(0, T)$, $0 < \beta_t < \bar{\beta}$. Thus, the HJB equation is satisfied if and only if $\dot{\alpha}_t - \mu\,\alpha_t = 0$, $1 - 2\lambda_t\,\alpha_t = 0$, and $\alpha_t(\sigma_v^2 + \lambda_t^2\sigma_y^2) + \dot{\gamma}_t - \mu\gamma_t = 0$. The first two conditions are met if and only if $\lambda_t = \lambda_0\,e^{-\mu t}$ and $\alpha_t = e^{\mu t}/[2\lambda_0]$, for some constant $\lambda_0 > 0$. Replacing these two functions, the solution of the last differential equation is

$$\gamma_t = \left[\gamma_0 - \frac{\lambda_0\sigma_y^2}{4\mu} - \alpha_0\sigma_v^2 t\right]e^{\mu t} + \frac{\lambda_0\sigma_y^2}{4\mu}e^{-\mu t},$$

for some constant γ_0.

In (T, ∞), $\beta_t \equiv \bar{\beta}$, so the HJB equation is satisfied if and only if $1 - 2\lambda_t\alpha_t \geq 0$,

$$\bar{\beta}(1 - 2\lambda_t\alpha_t) + \dot{\alpha}_t - \mu\,\alpha_t = 0 \qquad \text{and} \qquad \alpha_t\,\sigma_t^2 + \dot{\gamma}_t - \mu\gamma_t = 0.$$

We check the last two conditions first. The coefficient α_t defined by (8) satisfies

$$\dot{\alpha}_t = -(L-\mu)\,\alpha_t + \frac{2\,L\,e^{L\,t}}{e^{L\,t} - k\,e^{L\,T}}\,\alpha_t - \bar{\beta} = \mu\,\alpha_t + \bar{\beta}(2\,\lambda_t\,\alpha_t - 1). \tag{23}$$

The coefficient γ_t defined by (9) satisfies

$$\dot{\gamma}_t = \mu\,\gamma_t - \bar{\beta}\,e^{\mu\,t}\int_t^\infty e^{(L-\mu)\,s}[e^{L\,s} - k\,e^{L\,T}]^{-2}\sigma_t^2\,e^{-L\,t}[e^{L\,t} - k\,e^{L\,T}]^2\mathrm{d}s = \mu\,\gamma_t - \alpha_t\,\sigma_t^2. \tag{24}$$

Finally we show that $2\,\lambda_t\,\alpha_t < 1$ for all $t \in (T,\infty)$. Since $2\,\lambda_t\,\alpha_t = 1$ for all $t \in (0,T)$, we have that $2\,\lambda_T^-\,\alpha_T^- = 1$. The value-matching conditions (10) also require that $\lambda_T^- = \lambda_T$ and $\alpha_T^- = \alpha_T$. Therefore $2\,\lambda_T\,\alpha_T = 1$. To ensure the nonnegativity of λ_t for $t \geq T$, we must require that $|k| \leq 1$. Using the definitions of λ_t and α_t for $t \geq T$, and with the change of variable $u = L\,s$, it follows that

$$\begin{aligned} 2\,\alpha_t\,\lambda_t &= L\int_0^\infty e^{-(L+\mu)\,s}\,\frac{1-[k\,e^{-L(t-T)}]^2}{[1-k\,e^{-L(s+t-T)}]^2}\,\mathrm{d}s \\ &= \int_0^\infty e^{-(1+\mu/L)\,u}\,\frac{1-[k\,e^{-L(t-T)}]^2}{[1-k\,e^{-L(t-T)}\,e^{-u}]^2}\,\mathrm{d}u \;=\; h\big(k\,e^{-L(t-T)}\big). \end{aligned}$$

Note that the border condition $2\,\alpha_T\,\lambda_T = 1$ reduces to $h(k) = 1$ (a condition that is independent of T!). For notational convenience, let us define the constant $a = 1 + \mu/L$

In the argument that follows we will show that for $\bar{\beta}$ sufficiently large there exists a unique $k \in [-1,1]$ that solves $h(k) = 1$. Furthermore, we will show that $k > 0$ and that $h(\ell)$ is increasing in ℓ for $\ell \in [0,k]$. As a result, $2\,\alpha_t\,\lambda_t = h\big(k\,e^{-L(t-T)}\big) < h(k) = 1$ for all $t > T$, as required. Although the proof of these steps is tedious, the intuition is rather straightforward if we note that $h(\ell) \approx (1+\ell)/a$ for $\bar{\beta}$ sufficiently large (or equivalently, for a close to one).

To prove that $k > 0$ note that

$$h(0) = \frac{1}{a} < 1 \qquad \text{and} \qquad h'(\ell) = \int_0^\infty e^{-a\,u}\,\frac{2\,(e^{-u}-\ell)}{[1-\ell\,e^{-u}]^3}\,\mathrm{d}u,$$

so $h'(\ell) \geq 0$ for all $\ell \leq 0$. To prove the existence, we compute a lower bound for $h(\ell)$:

$$h(\ell) \geq \int_0^\infty e^{-a\,u}\,\frac{1-\ell^2}{[1-\ell\,e^{-a\,u}]^2}\,\mathrm{d}u = \frac{1+\ell}{a} \qquad \text{for all } \ell \in [0,1),$$

where the inequality follows from the fact that $a \geq 1$ and $\ell \geq 0$. This lower bound is equal to 1 at $\ell = a - 1 = \mu/L$. Hence, for $\bar{\beta}$ sufficiently large $\mu/L < 1$ and $h(\mu/L) > 1$. Since $h(\ell)$ is continuous and $h(0) = 1/a < 1$, there exists $k \in (0,\mu/L)$ such that $h(k) = 1$.

Assume that $\bar{\beta}$ is sufficiently large so that $\mu/L < 1$. We now conclude the proof by showing that $h(k)$ is increasing in $\ell \in [0,\mu/L]$ which shows, by virtue of the lower bound above, that k is unique and that $h(\ell) \leq h(k)$ for all $\ell \in [0, k)$, as needed. For this, note that for all $\ell \in [0, \mu/L]$

$$\begin{aligned} h'(\ell) &= \int_0^\infty e^{-a\,u}\,\frac{2\,(e^{-u}-\ell)}{(1-\ell\,e^{-u})^3}\,\mathrm{d}u \geq 2\int_0^\infty \frac{e^{-(a+1)\,u}}{(1-\ell\,e^{-\frac{a+1}{2}\,u})^3}\,\mathrm{d}u - 2\,\ell\int_0^\infty \frac{e^{-u}}{(1-\ell\,e^{-u})^3}\,\mathrm{d}u \\ &= \frac{1}{(1-\ell)^2}\left[\frac{2}{a+1} - (2-\ell)\,\ell\right] > \frac{5-3\,a^2}{(1-\ell)^2(a+1)}\,, \end{aligned}$$

where the last inequality uses the fact that $(2-\ell)\ell$ is increasing in $\ell \in [0, 1]$, so $(2-\ell)\ell \leq (3-a)(a-1)$, and the inequality $a(a^2-1) > 0$. Therefore, $h'(\ell) > 0$ for all $\bar{\beta}$ sufficiently large so that $a \leq \sqrt{5/3}$

To show that $\Pi(t, M) = \alpha_t M^2 + \gamma_t$ is continuously differentiable once in t and twice in M, note that the value matching conditions (10) imply that λ_t, α_t and γ_t are continuous functions of t. $\Pi(t, M)$ is clearly twice continuously differentiable in M. Hence, we need only show that α_t and γ_t are continuously differentiable at $t = T$. Equations (8), (23), (9) and (24) imply that

$$\lim_{t\uparrow T}\ \dot{\alpha}_t = \mu\,\alpha_T = \lim_{t\downarrow T}\ \dot{\alpha}_t \qquad \text{and} \qquad \lim_{t\uparrow T}\ \dot{\gamma}_t = \mu\,\gamma_T - \alpha_T\,\sigma_T^2 = \lim_{t\downarrow T}\ \dot{\gamma}_t.$$

To complete the proof one can show, after tedious computations, that β_t is increasing in $t \in [0, T]$. Since $\beta_T = \bar{\beta}$, the insider's strategy satisfies $|\beta_t| \leq \bar{\beta}$ for all t. $\blacksquare$

Proof of Theorem 2.

Assume that $\bar{\beta}$ is sufficiently large so that $\mu/L < 1/2$. By Theorem 1, $0 < k(\bar{\beta}) < \mu/L$. Hence

$$1 \leq \left[\frac{1+k(\bar{\beta})}{1-k(\bar{\beta})}\right] = \frac{1+\mu/L}{1-\mu/L} \leq 1 + 2\frac{\mu}{L} + 4\left[\frac{\mu}{L}\right]^2.$$

Since $L = 2\sigma_v\bar{\beta}/\sigma_y$, it follows that $k(\bar{\beta}) \to 0$ and so for all $t \geq T$,

$$\lim_{\bar{\beta}\to\infty} \lambda_t = \lim_{\bar{\beta}\to\infty} \frac{\sigma_v}{\sigma_y}\left[\frac{1+k(\bar{\beta})}{1-k(\bar{\beta})}\right] = \frac{\sigma_v}{\sigma_y} \qquad \text{and} \qquad \lim_{\bar{\beta}\to\infty} \Sigma_t = \lim_{\bar{\beta}\to\infty} \frac{\sigma_y^2\lambda_t}{\bar{\beta}} = 0.$$

Also, equation (8) implies that

$$\alpha_T = \lim_{\bar{\beta}\to\infty} \bar{\beta}\,e^{(L+\mu)\,T}\int_T^\infty e^{-(L+\mu)\,s}\,\mathrm{d}s = \lim_{\bar{\beta}\to\infty} \frac{\bar{\beta}}{L+\mu} = \frac{\sigma_y}{2\sigma_v}\,.$$

Therefore $\lambda_0 = e^{\mu T}/[2\alpha_T] = e^{\mu T}\sigma_v/\sigma_y$. In the limit, as $\bar{\beta} \to \infty$, the threshold time T solves

$$\Sigma_T = \Sigma_0 + \sigma_v^2\,T - \sigma_v^2\left[\frac{e^{2\,\mu\,T}-1}{2\,\mu}\right] = 0.$$

One can check that this equation has a unique solution and that $\dot{\Sigma}_T = 0$.

It only remains to prove the weak convergence of $M_t(\bar{\beta})$ to 0 as $\bar{\beta} \to \infty$. For this we will invoke Theorem 2.1 in Prokhorov (1956) and prove the convergence of the finite-dimensional distributions of $\{M_t(\bar{\beta})\}$ to 0 together with the compactness of the sequence $\{M_t(\bar{\beta})\}$ (see also Billingsley 1999, Chapter 2). Let $\mathcal{T} = [T_1, T_2]$ with $T < T_1 < T_2$. In what follows, we define $\Lambda(t,s) = \int_t^s \lambda_u \, \mathrm{d}u$, $\underline{\lambda}_{\mathcal{T}} = \min\{\lambda_t \,:\, t \in \mathcal{T}\}$, $\bar{\lambda}_{\mathcal{T}} = \max\{\lambda_t \,:\, t \in \mathcal{T}\}$ and $\bar{\sigma}_{\mathcal{T}} = \max\{\sigma_t \,:\, t \in \mathcal{T}\}$.

Let $\{t_1, t_2, \ldots, t_n\} \in \mathcal{T}$. For each $t \in \mathcal{T}$, $M_t(\bar{\beta})$ satisfies

$$M_t(\bar{\beta}) = M_T \, e^{-\bar{\beta}\,\Lambda(T,t)} + \int_T^t \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t)} \, \mathrm{d}B_s.$$

Therefore, the random vector $(M_{t_1}(\bar{\beta}), M_{t_2}(\bar{\beta}), \ldots, M_{t_n}(\bar{\beta}))$ has a Gaussian distribution. We now show that this distribution converges to the distribution of the constant vector $(0, \ldots, 0)$. Let us denote by $\mu^M(\bar{\beta})$ and $\Sigma^M(\bar{\beta})$ its mean vector and variance-covariance matrix, respectively. It follows that

$$\mu_i^M(\bar{\beta}) = \mathbb{E}[M_{t_i}(\bar{\beta})] = \mathbb{E}[M_T] \, e^{-\bar{\beta}\,\Lambda(T,t_i)}, \qquad i = 1, \ldots, n.$$

Similarly, the covariance between the i^{th} and j^{th} coordinates is (assume $t_i \leq t_j$)

$$\begin{aligned}
\Sigma_{ij}^M(\bar{\beta}) &= \mathbb{E}[\big(M_{t_i}(\bar{\beta}) - \mu_i^M(\bar{\beta})\big)\,\big(M_{t_j}(\bar{\beta}) - \mu_j^M(\bar{\beta})\big)] \\
&= \mathbb{E}\left[\left(\int_T^{t_i} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_i)} \, \mathrm{d}B_s\right) \left(\int_T^{t_j} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_j)} \, \mathrm{d}B_s\right)\right] \\
&= \mathbb{E}\left[\int_T^{t_i} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_i)} \, \mathrm{d}B_s \left(e^{-\bar{\beta}\,\Lambda(t_i,t_j)} \int_T^{t_i} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_i)} \, \mathrm{d}B_s + \int_{t_i}^{t_j} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_j)} \, \mathrm{d}B_s\right)\right] \\
&= e^{-\bar{\beta}\,\Lambda(t_i,t_j)} \, \mathbb{E}\left[\left(\int_T^{t_i} \sigma_s \, e^{-\bar{\beta}\,\Lambda(s,t_i)} \, \mathrm{d}B_s\right)^2\right] = e^{-\bar{\beta}\,\Lambda(t_i,t_j)} \left(\int_T^{t_i} \sigma_s \, e^{-2\,\bar{\beta}\,\Lambda(s,t_i)} \, \mathrm{d}s\right).
\end{aligned}$$

The fourth equality uses the fact that B_t has independent increment so that two stochastic integrals with non-overlapping ranges are uncorrelated. The fifth equality uses Itô's isometry. Therefore, as $\bar{\beta}$ goes to infinity we get

$$\lim_{\bar{\beta} \to \infty} \mu_i^M(\bar{\beta}) = 0 \qquad \text{and} \qquad \lim_{\bar{\beta} \to \infty} \Sigma_{ij}^M(\bar{\beta}) = 0, \qquad \text{for all } i, j = 1, \ldots, n.$$

We conclude that the distribution of $(M_{t_1}(\bar{\beta}), M_{t_2}(\bar{\beta}), \ldots, M_{t_n}(\bar{\beta}))$ converges to the distribution of the constant $(0, \ldots, 0)$.

We now prove that $\{M_t(\bar{\beta}) \,:\, \bar{\beta} > 0\}$ is tight. For this we show that there exists a constant R independent of $\bar{\beta}$ such that

$$D := \mathbb{E}\left[\big(M_{t_2}(\bar{\beta}) - M_{t_1}(\bar{\beta})\big)^2\right] \leq R\,|t_2 - t_1|, \qquad \text{for all } t_1, t_2 \in \mathcal{T}.$$

Indeed, by the definition of $M_t(\bar\beta)$ and Itô's isometry it follows for $t_1 \le t_2$ that

$$D = \left(M_T^2\, e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right)\left(1 - e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 + \int_{t_1}^{t_2} \sigma_s^2 e^{-2\bar\beta\Lambda(s,t_2)}\mathrm{d}s.$$

Since $(1-\exp(-x))^2 \le 2\,x$ for all $x \ge 0$, it follows that

$$\left(1 - e^{-\bar\beta\Lambda(t_1,t_2)}\right)^2 \le 2\,\bar\beta\Lambda(t_1,t_2) \le 2\,\bar\beta\,\bar\lambda_{\mathcal{T}}\,(t_2 - t_1).$$

As a result, we have that

$$D \le \left[2\,\bar\lambda_{\mathcal{T}}\left(M_T^2\,\bar\beta\, e^{-2\bar\beta\Lambda(T,t_1)} + \int_T^{t_1} \sigma_s^2\,\bar\beta\, e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s\right) + \bar\sigma_{\mathcal{T}}^2\right](t_2 - t_1).$$

Since $t_1 \ge T_1 > T$, it is not hard to show that

$$\bar\beta\, e^{-2\bar\beta\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,t_1)} \le \frac{e^{-1}}{2\,\Lambda(T,T_1)}.$$

In addition,

$$\int_T^{t_1} \sigma_s^2\,\bar\beta\, e^{-2\bar\beta\Lambda(s,t_1)}\mathrm{d}s \le \bar\sigma_{\mathcal{T}}^2 \int_T^{t_1} \bar\beta\, e^{-2\bar\beta\,\underline{\lambda}_{\mathcal{T}}\,(t_1-s)}\mathrm{d}s = \frac{\bar\sigma_{\mathcal{T}}^2}{2\,\underline{\lambda}_{\mathcal{T}}}.$$

Hence, we can choose the constant R to be equal to

$$R = \bar\lambda_{\mathcal{T}}\left(\frac{M_T^2\, e^{-1}}{\Lambda(T,T_1)} + \frac{\bar\sigma_{\mathcal{T}}^2}{\underline{\lambda}_{\mathcal{T}}}\right) + \bar\sigma_{\mathcal{T}}^2,$$

which is independent of $\bar\beta$. Hence, $\{M_t(\bar\beta)\}$ is tight. If $M_T = 0$ a.s. then we can repeat the previous steps with $T_1 = T$.

Finally, we prove the weak convergence of the insider's trading strategy. Let us denote by $X_t(\bar\beta)$ the insider's trading strategy when $\beta_t = \bar\beta$. Equation (2) implies that $X_t(\bar\beta)$ has the following dynamics

$$\mathrm{d}X_t(\bar\beta) = \bar\beta\, M_t(\bar\beta)\,\mathrm{d}t = \frac{1}{\lambda_t}\left[\sigma_v\,\mathrm{d}B_t^v - \lambda_t\,\sigma_y\,\mathrm{d}B_t^y - \mathrm{d}M_t(\bar\beta)\right], \qquad t \ge T.$$

Integrating from T to t we get $X_t(\bar\beta) = X_T + \sigma_y\,[(B_t^v - B_T^v) - (B_t^y - B_T^y)] - \frac{\sigma_y}{\sigma_v}\,(M_t(\bar\beta) - M_T)$. Since $M_T = 0$, $\lambda_t \to \sigma_v/\sigma_y$ and $M_t(\bar\beta)$ converges weakly to 0 as $\bar\beta \to \infty$ for all $t > T$, it follows that $X_t(\bar\beta) \overset{\bar\beta\to\infty}{\Longrightarrow} X_t = X_T + \sigma_y\,[(B_t^v - B_T^v) - (B_t^y - B_T^y)]$ for all $t \ge T$. ∎

Proof of Theorem 3. The insider's HJB optimality condition are given by

$$0 = \max_\beta\left\{-\lambda_t\,\beta\, M\,\Pi_M + \frac{1}{2}\,\lambda_t^2\,\sigma_y^2\,\Pi_{MM} + \Pi_t - \mu\,\Pi + M^2\,\beta\right\}, \qquad t \in [0,\infty).$$

Suppose, we guess a quadratic value function of the form $\Pi(t,M) = \alpha_t\, M^2 + \gamma_t$ for deterministic functions α_t and γ_t. The HJB equation is satisfied if and only if $\dot\alpha_t - \mu\,\alpha_t = 0$, $1 - 2\lambda_t\,\alpha_t = 0$, and $\alpha_t(\sigma_v^2(t) + \lambda_t^2\sigma_y^2) + \dot\gamma_t - \mu\gamma_t = 0$. The first two conditions lead to $\lambda_t = \lambda_0\, e^{-\mu t}$ and $\alpha_t = e^{\mu t}/[2\lambda_0]$, for some constant $\lambda_0 > 0$. Replacing these two functions, the solution of the last differential equation is

$$\gamma_t = \frac{1}{2\,\lambda_0}\,(C + \Gamma_t)\, e^{\mu\, t} + \frac{\sigma_y^2\,\lambda_t}{4\,\mu},$$

for some constant $C \geq 0$ (since $\Gamma_t \downarrow 0$ as $t \to \infty$, $C \geq 0$ is required to ensure that $\gamma_t \geq 0$ for all t).

Note that the HJB condition does not provide any information about how to select the insider's strategy β_t. (Effectively, we have solved the HJB using the fact that insider is indeed indifferent). To determine the value of β_t we must turn to the market maker's filtering conditions (see Proposition 1)

$$\Sigma_t\,\beta_t = \lambda_t\,\sigma_y^2 \qquad \text{and} \qquad \dot\Sigma_t = \sigma_v^2(t) - \sigma_y^2\,\lambda_t^2$$

that guarantee that the market maker equilibrium condition $P_t = \mathbb{E}[V_t|\mathcal{F}_t^M]$ is satisfied. Since $\lambda_t = \lambda_0\, e^{-\mu\, t}$ it follows that

$$\Sigma_t = \Sigma_0 + \Gamma_0 - \Gamma_t - \frac{\sigma_y^2\,\lambda_0^2}{2\,\mu}\,(1 - e^{-\mu\, t}) \quad \text{and} \quad \beta_t = \frac{\sigma_y^2\,\lambda_0\, e^{-\mu\, t}}{\Sigma_t}.$$

To complete the proof, we need to specify the values of the two constant λ_0 and C and verify that the proposed value function $\Pi(t,M) = \alpha_t\, M^2 + \gamma_t$ and trading strategy β_t effectively solve the insider's problem. This final step is achieved by imposing the transversality condition $\lim_{t\to\infty} e^{-\mu\, t}\mathbb{E}[\Pi(t, M_t)] = 0$ for β_t.

To avoid confusions, we now use β_t to denote the trading strategy in equation (12) and $\tilde\beta_t$ to denote an arbitrary policy in $\mathcal{B}$. To make explicit the dependence of M_t on a trading strategy $\{\tilde\beta_s : 0 \leq s \leq t\}$ we will use the notation $M_t(\tilde\beta)$.

Since $\Pi(t,M) = \alpha_t\, M^2 + \gamma_t$ satisfies the HJB for any strategy $\tilde\beta \in \mathcal{B}$, it follows that

$$\Pi(0, M_0) = \mathbb{E}\left[\int_0^t e^{-\mu\, s}\,\tilde\beta_s\, M_s^2(\tilde\beta)\,\mathrm{d}s + e^{-\mu\, t}\,\Pi(t, M_t(\tilde\beta))\right] \geq \mathbb{E}\left[\int_0^t e^{-\mu\, s}\,\tilde\beta_s\, M_s^2(\tilde\beta)\,\mathrm{d}s\right]. \quad (25)$$

In addition,

$$e^{-\mu\, t}\mathbb{E}[\Pi(t, M_t(\beta_t))] = \frac{1}{2\,\lambda_0}\left(\mathbb{E}[M_t^2(\beta)] + C + \Gamma_t + \frac{\sigma_y^2\,\lambda_0^2\, e^{-2\,\mu\, t}}{2\,\mu}\right).$$

Thus, the transversality condition holds only if $C = 0$ and $\lim_{t\to\infty}\mathbb{E}[M_t^2(\beta)] = 0$. We can show that

$$\mathbb{E}[M_t^2(\beta)] = M_0^2\, e^{-2\,\lambda_0 \int_0^t e^{-\mu\, s}\,\beta_s\,\mathrm{d}s} + \int_0^t e^{-2\,\lambda_0 \int_s^t e^{-\mu\, u}\,\beta_u\,\mathrm{d}u}\,(\sigma_v^2(s) + \sigma_y^2\,\lambda_0^2\, e^{-2\,\mu\, s})\,\mathrm{d}s.$$

Hence, $\lim_{t\to\infty} \mathbb{E}[M_t^2(\beta)] = 0$ if $\int_0^\infty e^{-\mu\, s}\, \beta_s \,\mathrm{d}s = \infty$. This last requirement together with the fact that $\beta_t = \sigma_y^2\, \lambda_0\, e^{-\mu\, t}/\Sigma_t$ imply that $\lim_{t\to\infty} \Sigma_t = 0$. Therefore

$$\lambda_0 = \sqrt{\frac{2\,\mu\,(\Sigma_0 + \Gamma_0)}{\sigma_y^2}}.$$

With these choices of λ_0 and C, the transversality condition is satisfied for β_t, and taking limits in equation (25) we get that

$$\Pi(0, M_0) = \mathbb{E}\left[\int_0^\infty e^{-\mu\, s}\, \beta_s\, M_s^2(\beta)\,\mathrm{d}s\right] \geq \mathbb{E}\left[\int_0^\infty e^{-\mu\, s}\, \tilde{\beta}_s\, M_s^2(\tilde{\beta})\,\mathrm{d}s\right], \quad \text{for all } \tilde{\beta} \in \mathcal{B}.$$

In the last step we used (11) and $\beta_t > 0$ for all t, and invoked the Lebesgue Convergence Theorem to interchange limits and expectations. ▮

Proof of Proposition 2. The insider's ex-ante expected rent is

$$\mathbb{E}[\Pi(t, M_t)] = \alpha_t\, \mathbb{E}[(V_t - p_t)^2] + \gamma_t = \alpha_t\, \Sigma_t + \gamma_t.$$

For $t \in [T, \infty)$, $\Sigma_t = 0$ and $\mathbb{E}[\Pi(t, M_t)] = \gamma_T$. For $t \in (0, T)$, $\dot{\alpha}_t = \mu\, \alpha_t$, $2\, \alpha_t\, \lambda_t = 1$, and

$$\dot{\Sigma}_t = \frac{\mathrm{d}\Sigma_t}{\mathrm{d}\alpha_t}\, \dot{\alpha}_t = \sigma_v^2 - \lambda_t^2\, \sigma_y^2 = \sigma_v^2 - \frac{\sigma_y^2}{4\, \alpha_t^2} \qquad \text{so} \qquad \frac{\mathrm{d}\Sigma_t}{\mathrm{d}\alpha_t} = \frac{\sigma_v^2}{\mu\, \alpha_t} - \frac{\sigma_y^2}{4\, \mu\, \alpha_t^3}\,.$$

Therefore,

$$\Sigma_t = \frac{\sigma_v^2}{\mu}\, \log(\alpha_t) + \frac{\sigma_y^2}{8\, \mu\, \alpha_t^2} + c_1$$

for some constant of integration c_1. Similarly

$$\dot{\gamma}_t = \frac{\mathrm{d}\gamma_t}{\mathrm{d}\alpha_t}\, \dot{\alpha}_t = \mu\, \gamma_t - \alpha_t(\sigma_v^2 + \lambda_t^2 \sigma_y^2) \qquad \text{so} \qquad \frac{\mathrm{d}\gamma_t}{\mathrm{d}\alpha_t} = \frac{\gamma_t}{\alpha_t} - \frac{1}{\mu}\left[\sigma_v^2 + \frac{\sigma_y^2}{4\, \alpha_t^2}\right].$$

Hence

$$\gamma_t = -\frac{\sigma_v^2}{\mu}\, \alpha_t\, \log(\alpha_t) + \frac{\sigma_y^2}{8\, \mu\, \alpha_t} + c_2 \alpha_t,$$

for some constant c_2. Thus

$$\alpha_t\, \Sigma_t + \gamma_t = \frac{\sigma_y^2}{4\, \mu\, \alpha_t} + (c_1 + c_2)\alpha_t.$$

Since $\alpha_T = \sigma_y/[2\sigma_v]$, $\Sigma_T = 0$ and $\gamma_T = \sigma_v\, \sigma_y/\mu$, we have that

$$\frac{\sigma_v\, \sigma_y}{\mu} = \alpha_T\, \Sigma_T + \gamma_T = \frac{\sigma_y^2}{4\, \mu \alpha_T} + (c_c + c_2)\alpha_T,$$

which implies that $c_1 + c_2 = \sigma_v^2/\mu$. Therefore

$$\alpha_t\,\Sigma_t + \gamma_t = \frac{\sigma_y^2}{4\,\mu\,\alpha_t} + \frac{\sigma_v^2}{\mu}\,\alpha_t = \frac{\sigma_y\,\sigma_v}{2\,\mu}\,e^{\mu(T-t)} + \frac{\sigma_y\,\sigma_v}{2\,\mu}\,e^{-\mu(T-t)} = \frac{\sigma_y\,\sigma_v}{\mu}\,\cosh(\mu(T-t)). \qquad \blacksquare$$

Proof of Proposition 3. Recall from Theorem 2 that Σ_t satisfies

$$\Sigma_t = \Sigma_0 + \sigma_v^2\,t - \sigma_v^2\,e^{2\mu T}\,\left[\frac{1 - e^{-2\,\mu\,t}}{2\mu}\right] \qquad t < T,$$

and $\Sigma_T = 0$ for all $t \geq T$, where $T \geq 0$ is the unique solution to

$$\Sigma_0 + \sigma_v^2\,T = \sigma_v^2\,\left[\frac{e^{2\,\mu\,T} - 1}{2\,\mu}\right].$$

Since T decreases with σ_v, it suffices to prove that Σ_t decreases with σ_v for $t < T$.

In what follows, and without lost of generality, we will normalize the value of μ such that $2\,\mu = 1$ (this is equivalent to re-scaling time). With this normalization, the derivative of Σ_t $(t < T)$ with respect to σ_v^2 is equal to

$$\frac{\partial\Sigma_t}{\partial\sigma_v^2} = t - e^T\,(1 - e^{-t}) - \sigma_v^2\,e^T\,(1 - e^{-t})\,\frac{\partial T}{\partial\sigma_v^2}, \qquad t < T.$$

In addition, from the definition of T it follows that

$$\frac{\partial T}{\partial\sigma_v^2} = \frac{1}{\sigma_v^2}\,\left[\frac{1 + T - e^T}{e^T - 1}\right].$$

Plugging back this value on $\frac{\partial\Sigma_t}{\partial\sigma_v^2}$ we get that for $t < T$

$$\frac{\partial\Sigma_t}{\partial\sigma_v^2} = t - (1 - e^{-t})\,\left[\frac{T}{1 - e^{-T}}\right] \leq 0.$$

The inequality follows from the fact that $t/(1 - e^{-t})$ is an increasing function of t.

Let us now prove the monotonicity of the insider's ex-ante expected payoff. Given the normalization $2\,\mu = 1$, this payoff is given by

$$\mathbb{E}[\Pi(t, M_t)] = 2\,\sigma_y\,\sigma_v\,\cosh\left(\frac{1}{2}\,(T - t)^+\right) \qquad t \geq 0.$$

Note that to prove the monotonicity of $\mathbb{E}[\Pi(t, M_t)]$ with respect to σ_v it is enough to focus on the case $t \leq T$. The derivative with respect to σ_v is given by

$$\begin{aligned}
\frac{\partial\mathbb{E}[\Pi(t, M_t)]}{\partial\sigma_v} &= 2\,\sigma_y\,\cosh\left(\frac{1}{2}\,(T - t)\right) + \sigma_y\,\sigma_v\,\sinh\left(\frac{1}{2}\,(T - t)\right)\,\frac{\partial T}{\partial\sigma_v} \\
&= 2\,\sigma_y\,\cosh\left(\frac{1}{2}\,(T - t)\right) + 2\,\sigma_y\,\sinh\left(\frac{1}{2}\,(T - t)\right)\,\left[\frac{1 + T - e^T}{e^T - 1}\right] \\
&= 2\,\sigma_y\,\sinh\left(\frac{1}{2}\,(T - t)\right)\,\left[\frac{T}{e^T - 1}\right] + 2\,\sigma_y\,\exp\left(\frac{T - t}{2}\right) \geq 0. \qquad \blacksquare
\end{aligned}$$

References

Amihud, Y., H. Mendelson, L.H. Pedersen. 2006. Liquidity and asset prices. *Foundations and Trends in Finance* **1** 269–364.

Back, K. 1992. Insider trading in continuous time. *Review of Financial Studies* **5** 387–409.

Back, K., S. Baruch. 2004. Information in securities markets: Kyle meets Glosten and Milgrom. *Econometrica* **72**(2) 433–465.

Back, K., H. Pedersen. 1998. Long-lived information and intraday patterns. *Journal of Financial Markets* **1** 385–402.

Bagehot, W. 1971. The only game in town. *Financial Analyst Journal* **22** 12–14.

Biais, B., L. Glosten, C. Spatt. 2005. Market microstructure: A survey of microfundations, empirical results and policy implications. *Journal of Financial Markets* **8** 217–264.

Billingsley, P. 1999. *Convergence of Probability Measures*. 2nd ed. John Wiley & Sons, Inc., New York.

Brunnermeier, M.K. 2001. *Asset Pricing under Asymmetric Information*. Oxford University Press, New York.

Caldentey, R., E. Stacchetti. 2007. Insider tarding with stochastic valuation. Available at SSRN: http://ssrn.com/abstract=986405.

Chau, M., D. Vayanos. 2006. Strong-form efficiency with monopolistic insiders. Tech. rep., London School of Economics.

Dasgupta, A., A. Prat. 2005. Reputation and asset prices: A theory of information cascades and systematic mispricing. Tech. rep., London Business School.

Easley, D., M. O'Hara. 1987. Price, trade size, and information in securities markets. *Journal of Financial Economics* **19** 69–90.

Foster, F.D., S. Viswanathan. 1996. Strategic trading when agents forecast the forecasts of others. *The Journal of Finance* **LI**(4) 1437–1478.

Fudenberg, D., D.K. Levine. 2007. Repeated games with frequent signals, mimeo.

Glosten, L. 1989. Insider trading, liquidity and the role of monopolist specialist. *Journal of Business* **62** 211–235.

Glosten, L., P. Milgrom. 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics* **14** 71–100.

Holden, G.W., A. Subrahmanyam. 1992. Long-lived private information and imperfect competition. *The Journal of Finance* **XLVII**(1) 247–270.

Kyle, A. 1985. Continuous auctions and insider trading. *Econometrica* **53** 1315–1335.

Mendelson, H., T. Tunca. 2004. Strategic trading, liquidity, and information acquisition. *Review of Financial Studies* **17** 295–337.

O'Hara, M. 1997. *Market Microstucture Theory*. Blackwell Publishing, Massachusetts.

Prokhorov, Y. 1956. Convergence of random processes and limit theorems in probability theory. *Theor. Prob. and Appl.* **1**(2) 157–214.

Spiegel, M., A. Subrahmanyam. 1992. Informed speculation and hedging in a noncompetitive securities market. *Review of Financial Studies* **5** 307–329.