EC 147: Bargaining Theory and Applications
Solutions to Homework 1

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1. (a) The strategic or normal form of a game consists of three elements: players, strategies, and payoffs. In this game, there are two players: John and Mary. Each player has two strategies: football (F) and opera (O). The payoffs are shown in the following matrix:

	F	O
F	2,1	0,0
O	0,0	1,2

where the rows correspond to John's choices while columns refer to Mary's choices.

(b) There are two pure strategy Nash equilibria: (F, F) and (O, O)

(c) Let p be the probability that John plays F and q the probability that Mary plays F. The indifference conditions are:

$$1p + 0(1-p) = 0p + 2(1-p)$$
$$2q + 0(1-q) = 0q + (1-q)$$

Solving these equations, we find that the mixed strategy equilibrium is $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$, where the first pair is the way John randomizes in equilibrium, whereas the second pair is Mary's equilibrium mixed strategy.

2.

a. The game has two pure strategy Nash equilibria, (U, L) and (D, R). Note that if you tried to compute the mixed strategy equilibrium, you'd get the pure strategy ones.

b. The game has two pure strategy Nash equilibria, (U, L) and (D, R), and one mixed strategy equilibrium $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$, where the first and second pairs are Player 1's and Player 2's equilibrium mixed strategies, respectively.

c. The game has three pure strategy Nash equilibria, (U, L), (M, C), and (L, R), and one mixed strategy equilibrium $\left(\left(0, \frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{2}{3}, \frac{1}{3}\right)\right)$, where the first and second triads are Player 1's and Player 2's equilibrium mixed strategies, respectively.

d. The game has two pure strategy Nash equilibria, (U, L) and (D, R), and a continuum of mixed strategy equilibria $\Big(\big(\lambda, 1-\lambda\big), \big(0, 1\big)\Big)$, with $\lambda \in (0, 2/3)$. Again, the first and second pairs are Player 1's and Player 2's equilibrium mixed strategies, respectively.

3.

a. Firm i takes q_j as given and solves (for simplicity, we'll deal only with interior solutions):

$$\text{Max}_{\{q_i\}} \;\; \pi_i = p\; q_i - (q_i + q_j)\; q_i$$

The solution to this program gives us the best response function

$$q_i(q_j) = \frac{p - q_j}{2}$$

That is, $q_1(q_2) = \frac{p-q_2}{2}$ and $q_2(q_1) = \frac{p-q_1}{2}$.
Solving this system gives the Nash equilibrium $(q_1^{NE}, q_2^{NE}) = \left(\frac{p}{3}, \frac{p}{3}\right)$.
The respective profits are $\pi_1^{NE} = \pi_2^{NE} = \frac{p^2}{9}$.

b. The newly formed firm solves

$$\text{Max}_{\{q\}} \;\; \pi = p\; q - q^2$$

Hence, the optimal level of output is $q^M = \frac{p}{2}$, which yields a profit $\pi^M = \frac{p^2}{4}$.

c. Note that $q_i^{NE} < q^M < \sum_{i=1}^{2} q_i^{NE}$ and $\pi^M > \pi_2^{NE}$. The reason of this result stems from the fact that the unified firm takes into consideration the negative externality it generates (i.e., it internalizes the externality). Thus, it produces less than the aggregate production in the decentralized case, which yields higher profits.

27.1.

a. The payoff matrix with altruism becomes:

	Q	F
Q	$2+2\alpha$, $2+2\alpha$	3α, 3
F	3, 3α	$1+\alpha$, $1+\alpha$

If $\alpha = 1$, the payoff matrix becomes:

	Q	F
Q	4,4	3,3
F	3,3	2,2

This game is no longer the Prisoner's Dilemma, as the Pareto efficient result (Q, Q) is the unique Nash equilibrium. Further, for each player, Q is the strictly dominant strategy.

b. For the game to continue being the Prisoner's Dilemma, we need three conditions, namely: (i) (Q, Q) Pareto dominates (F, F); (ii) F is the best response to Q (i.e., there are no incentives to cooperate); (iii) F is the best response to F, so that (F, F) becomes a NE as well. Condition (i) follows immediately $\forall \alpha > 0$. Condition (ii) implies that $3 > 2 + 2\alpha$, whereas (iii) requires $1 + \alpha > 3\alpha$. Both conditions are met if $\alpha < 1/2$. In sum, we need

$\alpha \in (0, 1/2)$.
If $\alpha = 1/2$, the payoff matrix looks like:

	Q	F
Q	3,3	1.5,3
F	3,1.5	1.5,1.5

In this game, there are four Nash equilibria $(Q, Q), (Q, F), (F, Q)$ and (F, F).
Finally, for $\alpha > 1/2$, the unique Nash equilibrium is (Q, Q).

27.2 (a) We can model the game with the following payoff matrix:

	Sit	Stand
Sit	v_{ss}, v_{ss}	$v_s, 0$
Stand	$0, v_s$	$0, 0$

where $v_s > v_{ss} > 0$. This game is not the Prisoner's Dilemma, as (Stand, Stand) is not an equilibrium. The unique Nash equilibrium is (Sit, Sit).

(b) We can model the game with the following payoff matrix:

	Sit	Stand
Sit	v_{ss}, v_{ss}	$0, v_s$
Stand	$v_s, 0$	v_t, v_t

where $v_s > v_{ss} > v_t > 0$. This is the Prisoner's Dilemma, whose unique Nash equilibrium is (Stand, Stand).

(c) Both individuals travel more comfortable when they play selfishly.

34.2

a. If there are only one supporter for each candidate the game is equivalent to a prisoners' dilemma game. Both have an incentive to vote and they end up in a tie as if they had not voted but having paid the cost of voting.

b. With an equal number of supporters for each candidate the unique NE is for all supporters to vote.

c. There is no NE in pure strategies. See problem 118.2 for the NE in mixed strategies.

34.3. As in the first problem, we have to define the elements of the game in its normal form. There are four players. Each of them has two strategies (the route via X and the route via Y). Their respective payoff is negatively related to the time they spent in the commute (i.e., they prefer to get to B as quick as possible).
In order to find the Nash equilibria, we can start by assuming that all four people will drive on one of the routes, say A-X-B, and see if it is profitable for any of the drivers to deviate and take the alternative route. We will proceed this way until we find a situation where no driver seeks to deviate. If all four take the A-X-B route, each spends 15 + 22.7 = 37.7 minutes in

the commute. If one person deviates and takes the A-Y-B route, she will travel for 20 + 6 = 26 minutes, which makes her better off, so she'll deviate. Will another person deviate? Yes, because two people taking the A-Y-B route will commute for 9 + 21 = 30 minutes, as opposed to having three people on the A-X-B route commuting for 12 + 21.8 = 33.8 minutes each.
At this point we have two people taking the A-Y-B route who travel for 30 minutes, while the other two take the A-X-B route and travel for 9 + 20.9 = 29.9 minutes. Now, there are no incentives to deviate since doing so would mean a longer commute. Therefore, the set of Nash equilibria consists of those profiles in which two people take the A-X-B route and two people take the A-Y-B route.
A similar logic applies to the case with the additional route. Let's start by assuming that all four people will drive on the new route A-X-Y-B (they all are excited about testing the new option). In this scenario, each person commutes for 15 + 10 + 15 = 40 minutes. Note that even in this case, there are incentives to take the A-Y-X-B route since doing so would imply a 20 + 7 + 20 = 47 minute commute. Hence, in equilibrium, no one takes that route. There will be incentives for two people to take the A-X-B and A-Y-B routes separately. The former represents a 12 + 20 = 32 minute commute, while the latter implies a 20 + 12 = 32 minute commute as well. The remaining two people taking the A-X-Y-B route commute for 12 + 8 + 12 = 32 minutes. In this context, there are no further incentives to deviate. Therefore, the set of Nash equilibria consists of those profiles in which one driver takes the A-Y-B route, two the A-X-Y-B route and one the A-X-B route. Each driver commutes for 32 minutes, which is worse than the equilibrium situation without the added road.

114.3. The game has two pure strategy equilibria, (NE, NE) and (E, E), and one mixed strategy equilibrium $((1-c, c), (1-c, c))$.[1] A change in c does not affect the players' ranking of the four outcomes, so the pure strategy equilibria are not affected. However, since the probability that players will exert effort is precisely c, an increase of this variable increases the probability that players will coordinate.

118.2. In equilibrium $p = c^{\frac{1}{k-1}}$.

This probability is such that supporters of candidate A are indifferent between voting or abstaining. The utility of not voting is zero while the utility of voting is $p^{k-1} - c$. The equilibrium p is obtained from solving the equation $p^{k-1} - c = 0$.

This probability has to be such that the k B supporters who are supposed to vote want to vote, and the $m - k$ who are suppose to abstain want to abstain.

For a B voter who is suppose to vote the expected utility of voting is equal to $2\left(1-p^k\right) + p^k - c$ and the utility of abstaining is $kp^{k-1}(1-p) + 2(1 - p^k - kp^{k-1}(1-p))$. For these B voters to be willing to vote it must be the case that the former expected utility is greater or equal than the latter. This implies that $p^k + kp^{k-1}(1-p) \geq c$.

As in equilibrium $p = c^{\frac{1}{k-1}}$, this condition is satisfied as $c^{\frac{k}{k-1}} + kc(1 - c^{\frac{1}{k-1}}) \geq c^{\frac{k}{k-1}} + c(1 - c^{\frac{1}{k-1}}) = c$ given that $k \geq 1$.

[1] Notice that this is equivalent to saying that there are three mixed strategy equilibria $((1,0),(1,0))$, $((0,1),(0,1))$ and $((1-c,c),(1-c,c))$.

For a B voter who is suppose to abstain the expected utility of voting is equal to $2 - c$ and the utility of abstaining is $2(1 - p^k) + p^k$. For these B voters to be willing to abstain it must be the case that the former expected utility is lower or equal than the latter. It necessary then that $p^k \leq c$ which holds under the equilibrium probability since $c^{\frac{k}{k-1}} < c$ by the fact that $\frac{k}{k-1} > 1$ and $c < 1$.