EC 147: Bargaining Theory and Applications
Solutions to Homework 2

Department of Economics
Brown University

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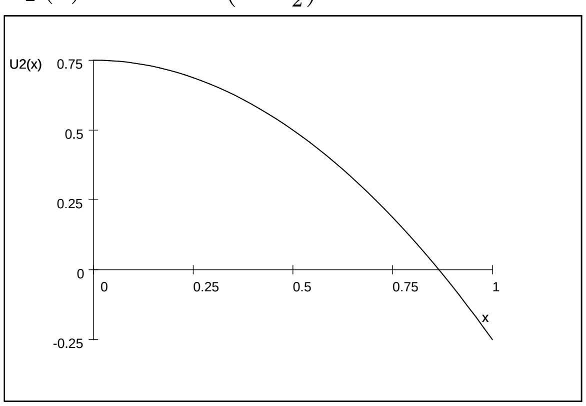
1. (a) Recall, a player's strategy is a complete contingent plan (or decision rule) that specifies how the player will act in each information set. In this game, P2's set of strategies is defined by some function $f : [0,1] \to \{\text{Accept, Reject}\}$ that determines whether P2 accepts or rejects each possible offer made by P1.

   (b) P2's best response to each offer of P1 is to Accept, as $1 - x \geq 0,\ \forall x$. This also applies to the case in which $x = 1$; otherwise, there would be no best offer for P1 and so there will be no SPNE. P1 is aware of P2's strategy, so he sets $x = 1$. The SPNE is therefore, $x = 1$, while P2 accepts any offer.

   (c) Yes, there is. It is given by P2 accepting only the offer $x = 0$ and rejecting all other offers and P1 in turn making the offer $x = 0$. Note, however, that this NE involves a non-credible threat by P2, so it wouldn't be a SPNE.

   (d) $U_2(x) = 1 - x - \left(x - \frac{1}{2}\right)^2$



   (e) P2 accepts any offer as long as $U_2(x) \geq 0$; that is, if $x \leq \frac{\sqrt{3}}{2}$. As P1 wants his share to be as big as possible but without violating P2's condition to accept. [1] Hence, in equilibrium, the condition is binding, so $x^{SPNE} = \frac{\sqrt{3}}{2}$, while P2 accepts this offer.

   (f) The equality-caring player.

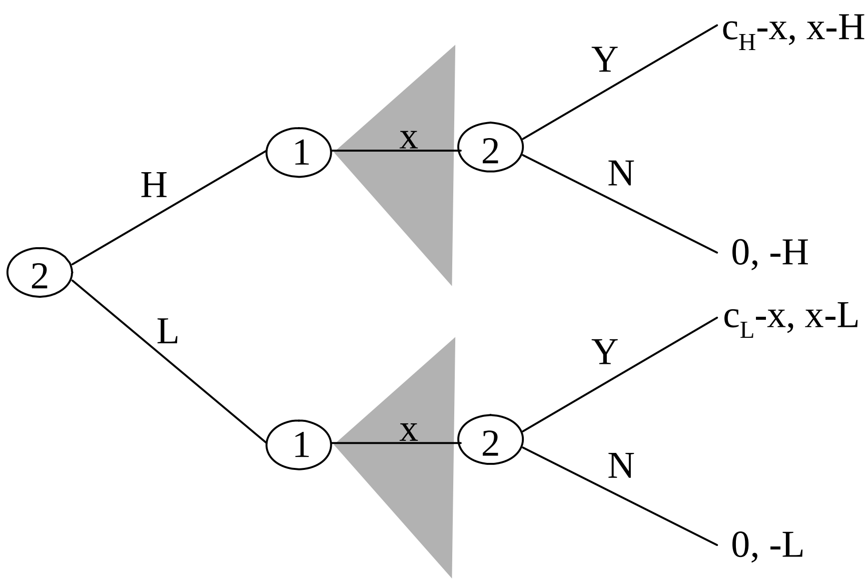
[1] In the economic jargon this condition is called "incentive compatibility."

185.1. P1's strategies are $s_1 = \{\text{Greedy, Equal, Nice}\}$. P2 has 8 strategies, which are $(R, R, R), (R, R, A), (R, A, R), (A, R, R), (R, A, A), (A, A, R), (A, R, A), (A, A, A)$, where R stands for Reject and A for accept.
There are two subgame perfect Nash equilibria: $(G, (A, A, A))$ and $(E, (R, A, A))$. To find the other NE, we can express the strategic for of the game as follows:

	R,R,R	R,R,A	R,A,R	A,R,R	R,A,A	A,A,R	A,R,A	A,A,A
G	$0,0$	$0,0$	$0,0$	$2,0$	$0,0$	$2,0$	$2,0$	$2,0$
E	$0,0$	$0,0$	$1,1$	$0,0$	$1,1$	$1,1$	$0,0$	$1,1$
N	$0,0$	$0,2$	$0,0$	$0,0$	$0,2$	$0,0$	$0,2$	$0,2$

The Nash equilibria are $(G, (R, R, R)), (G, (R, R, A)), (G, (A, R, R)), (G, (A, A, R)), (G, (A, R, A)), (G, (A, A, A)), (E, (R, A, R)), (E, (R, A, A))$ and $(N, (R, R, A))$. Thus, the only profiles that are both a NE and a SPNE are $(G, (A, A, A))$ and $(E, (R, A, A))$.
The subgame perfect Nash equilibria $(G, (A, A, A))$ and $(E, (R, A, A))$ result in the division $(2, 0)$ and $(1, 1)$, respectively. The division $(0, 0)$ can be generated by the Nash equilibria $(G, (R, R, R))$ and $(G, (R, R, A))$, whereas the outcome $(0, 2)$ can be generated by the Nash equilibrium $(N, (R, R, A))$. However, none of these equilibria are SPNE.

185.2.a. Unlike in problem 1, player 2 has some bargaining power, in fact equal bargaining power! Let the cake be of size C and $(x, C - x)$ be the division that player 1 proposes. Now, player 2's best response is to simply select the bigger of the two pieces. So in the unique SPNE, it is true that $x = C - x \Rightarrow x^{SPNE} = \frac{C}{2}$. That is, player 1 divides the cake equally and both players end up with equal shares.



186.1

189.1. Let's assume that firm 1 is the leader and firm 2 the follower. To solve this problem, we proceed by backward induction. In the second stage of the game, firm 2 solves

$$\text{Max}_{\{q_2>0\}} \quad \pi_2 = (\alpha - q_1 - q_2)q_2 - q_2^2$$

The solution to this program gives us the best response function

$$q_2^{BR}(q_1) = \frac{\alpha - q_1}{4}$$

In the first stage of the game, firm 1 solves

$$\text{Max}_{\{q_1>0\}} \ \ \pi_1 = \Big(\alpha - q_1 - q_2^{BR}(q_1)\Big)q_1 - q_1^2 = \Big(\alpha - q_1 - \frac{\alpha - q_1}{4}\Big)q_1 - q_1^2$$

The solution to this program yields the Stackelberg equilibrium output for firm 1 (the leader), $q_1^S = \frac{3}{14}\alpha$. It follows that the equilibrium output for firm 2 (the follower) is $q_2^S = q_2^{BR}(q_1^S) = \frac{11}{56}\alpha$. Total output is therefore $Q^S = q_1^S + q_2^S = \frac{23}{56}\alpha$. Finally, the equilibrium profits are $\pi_1^S = \frac{63}{784}\alpha^2$ and $\pi_2^S = \frac{60.5}{784}\alpha^2$.
Notice that $q_1^S > q_2^S$ and $\pi_1^S > \pi_2^S$. This results from what is often called "first mover advantage."
In the Cournot game, the best response functions are

$$q_1^{BR}(q_2) = \frac{\alpha - q_2}{4} \quad \text{and} \quad q_2^{BR}(q_1) = \frac{\alpha - q_1}{4}$$

Solving this system of equations gives the Cournot equilibrium output for both firms, $q_1^C = q_2^C = \frac{1}{5}\alpha$. Thus, $Q^C = \frac{2}{5}\alpha$. The equilibrium profits are $\pi_1^C = \pi_2^C = \frac{2}{25}\alpha^2$.
Comparing the Stackelberg and Cournot equilibrium results, we can see that $q_1^S > q_1^C$, $q_2^S < q_2^C$, $Q^S > Q^C$, $\pi_1^S > \pi_1^C$ and $\pi_2^S < \pi_2^C$.

233.1 If K=1 there are two possible NE: (In, Acquiesce) and (Out, Fight). Therefore either In happens and it is followed by Acquiesce or Out happens. For K>1, the same is true, in each period either In happens and it is followed by Acquiesce or Out happens.

429.1 Consider the following Grim Strategy:

$$\text{for } i = 1, 2, \quad s_i = \begin{cases} C & \text{if always } C \\ D & \text{otherwise} \end{cases}$$

Given that the other player is playing Grim and that player i will follow Grim in the future, the present discounted value of the stream of payoffs for player i if the players play (C, C) is

$$U_i(C|\text{Grim, Grim}) = \sum_{t=0}^{\infty} \delta^t x = \frac{x}{1-\delta}$$

If, however, player i deviates and plays D while the other player continues playing C, she gets

$$U_i(D|\text{Grim, Grim}) = y + \sum_{t=1}^{\infty} \delta^t 1 = y + \frac{\delta}{1-\delta}$$

There are no incentives to deviate if $U_i(C|\text{Grim, Grim}) \geq U_i(D|\text{Grim, Grim})$ which holds if $\delta \geq \frac{y-x}{y-1}$. Therefore, for (Grim, Grim) is a NE if $\delta \geq \frac{y-x}{y-1}$.
As the punishment is credible (if you expect the other player to defect forever, then defecting forever is optimal) the minimum discount factor needed for sub-game perfection is also $\delta \geq \frac{y-x}{y-1}$.

3. Repetition of the same game played by the same players allows them to design a mechanism that rewards cooperation (C) and punishes non-cooperation (D). One of such mechanisms is called the "Grim Strategy." For simplicity, we will assume that the game is first played in $t = 0$.

   (a) Consider the following Grim Strategy:

$$\text{for } i = 1, 2, \quad s_i = \begin{cases} C & \text{if always } C \\ D & \text{otherwise} \end{cases}$$

   Given that the other player is playing Grim and that player i will follow Grim in the future, the present discounted value of the stream of payoffs for player i if the player plays C is

$$U_i(C|\text{Grim, Grim}) = \sum_{t=0}^{\infty} \delta^t a = \frac{a}{1-\delta}$$

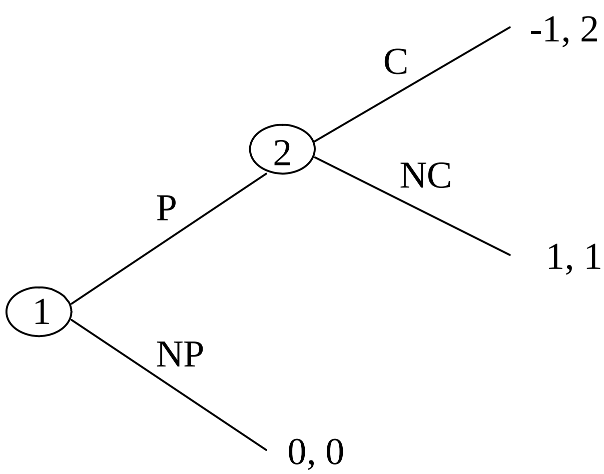
   If, however, player i deviates and plays D, she gets

$$U_i(D|\text{Grim, Grim}) = 4$$

   (C, C) can be supported as a SPNE if $U_i(C, C|\text{Grim}) \geq U_i(D, C|\text{Grim})$. That is, if $\delta \geq 1 - \frac{a}{4}$.

   (b) The critical level of δ is decreasing in a. The reason is that an increase in a means that the temptation to defect reduces (or another way to put it is that the cost to deviating is higher) and so even if players are more impatient, they would still prefer to play (C, C) forever rather than defecting today and getting nothing from tomorrow onwards.

4.



a.

b. Players = {Buyer, Seller}. Their strategies are $S_B = \{P, NP\}$ and $S_F = \{C, NC\}$, where P stands for Purchase, NP for No Purchase, C for Cheat and NC for No Cheat. The strategic form of the game looks like

	C	NC
P	$-1, 2$	$1, 1$
NP	$0, 0$	$0, 0$

The unique NE is (NP, C).

c. (NP, C) is also the SPNE of the game.

d. Consider the following grim strategy by the buyer:

$$s_B = \begin{cases} P & \text{if always } NC \\ NP & \text{otherwise} \end{cases}$$

The present discounted value of the stream of payoffs for the firm if the players play (P, NC) is $U_F(P, NC|\text{Grim}) = \sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$. If, however, the firm deviates and plays C while the other player continues playing P, it gets $U_F(P, C|\text{Grim}) = 2$. The firm does not cheat if $U_F(P, NC|\text{Grim}) \geq U_F(P, C|\text{Grim})$. That is, if $\delta \geq \frac{1}{2}$.

e. In this case, the resent discounted value of the stream of payoffs for the firm if the players play (P, NC) is $U_F(P, NC|\text{Grim}) = 1+\delta^{10}+\delta^{20}+\delta^{30}+\ldots = (\delta^{10})^0+(\delta^{10})^1+(\delta^{10})^2+(\delta^{10})^3+\ldots = \sum_{t=0}^{\infty}(\delta^{10})^t = \frac{1}{1-\delta^{10}}$. As before, if the firm deviates, it gets $U_F(P, C|\text{Grim}) = 2$. The firm does not cheat if $U_F(P, NC|\text{Grim}) \geq U_F(P, C|\text{Grim})$. That is, if $\delta \geq \left(\frac{1}{2}\right)^{\frac{1}{10}}$.

f. Note that $\left(\frac{1}{2}\right)^{\frac{1}{10}} > \frac{1}{2}$. That is, the requirement on δ is harder to satisfy if players interact only occasionally. Thus, we should trust someone with whom we play more often, which is the case of a restaurant.