Evolutionary Game Theory

+Learning: short-sighted agents that choose based on past experience.

+Evolution: strategies that are most successful grow faster.

Dynamic and Static concepts!

Static: Evolutionary Stable Strategies

Idea: a strategy is ESS if it repeals small invasions.

Example:

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

D is ESS, C is not. Regardless of the proportions of C and D, $U(C) < U(D)$. Therefore any small invasion of D would take over a population of all C. Any invasion of C would not take over a population of all D.

Evolutionary Stable Strategies

Definition: A strategy σ is ESS if

$U(\sigma, \varepsilon\sigma' + (1-\varepsilon)\sigma) > U(\sigma', \varepsilon\sigma' + (1-\varepsilon)\sigma)$ for all σ', for all $\varepsilon > 0$.

Proposition 1: A strategy σ is ESS if and only if

$$\begin{matrix} U(\sigma,\sigma) > U(\sigma',\sigma) \text{ or} \\ U(\sigma,\sigma) = U(\sigma',\sigma) \text{ and } U(\sigma,\sigma') > U(\sigma',\sigma') \end{matrix} \quad \text{for all } \sigma'.$$

Proof:

A strategy σ is ESS if

$U(\sigma, \varepsilon\sigma' + (1-\varepsilon)\sigma) > U(\sigma', \varepsilon\sigma' + (1-\varepsilon)\sigma)$ for all σ', for all $\varepsilon > 0$.

Then,

$\varepsilon U(\sigma,\sigma') + (1-\varepsilon) U(\sigma,\sigma) > \varepsilon U(\sigma',\sigma') + (1-\varepsilon) U(\sigma',\sigma)$ for all σ', for all $\varepsilon > 0$.■

Evolutionary Stable Strategies

Proposition 2: Every ESS is a NE.

Proof:

From proposition 1, if σ is ESS $U(\sigma, \sigma) \geq U(\sigma', \sigma)$ ■

Proposition 3: Every strict NE is an ESS.

Proof:

If σ is SNE then $U(\sigma, \sigma) > U(\sigma', \sigma)$ and σ is ESS by Proposition 1.

■

Evolutionary Stable Strategies

Example 1 of NE that is not ESS:

	A	B
A	0, 0	0, 0
B	0, 0	1, 1

A is a NE but is not ESS.

Example 2 of NE that is not ESS:

	A	B
A	2, 2	-100, 0
B	0, -100	1, 1

$\left(\frac{101}{103}A + \frac{2}{103}B, \frac{101}{103}A + \frac{2}{103}B\right)$ is NE but it is not ESS.

Evolutionary Stable Strategies

Proposition 4: While mixed strategy equilibria can never be strict, it can however be ESS.

Proof by example:

	A	B
A	0, 0	1, 1
B	1, 1	0, 0

$\left(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}A + \frac{1}{2}B\right)$ is mixed NE and ESS.

Invasion of ε of A, then:

$U(\frac{1}{2}A + \frac{1}{2}B) = (1 - \varepsilon)\,\frac{1}{2} + \varepsilon\frac{1}{2}$

$U(A) = (1 - \varepsilon)\,\frac{1}{2} + \varepsilon 0$

Then $U(\frac{1}{2}A + \frac{1}{2}B) > U(A)$.

Similarly for an invasion of B.■

Replicator Dynamics:

Explicit model of population dynamics.

Consider a single homogeneous population playing a symmetric game.

Continuous time.

$\phi_t(s)$ is the size of the population playing strategy s at time t.

Main Assumption:

$$\dot{\phi}_t(s) = \phi_t(s) u_t(s)$$

Net reproduction rate is proportional to payoff in stage game.

Replicator Dynamics:

$\theta_t(s)$ is the proportion playing strategy s at time t.

$$\dot{\theta}_t(s) = \theta_t(s)\left(u_t(s) - \overline{u}_t\right)$$

Reason:

$$\theta_t(s) = \frac{\phi_t(s)}{\sum \phi_t(s)}$$

$$u_t(s) = \sum \theta_t(s')u(s,s')$$

$$\overline{u}_t = \sum \theta_t(s)u_t(s)$$

$$\dot{\theta}_t(s) = \frac{\dot{\phi}_t(s)\sum \phi_t(s') - \phi_t(s)\sum \dot{\phi}_t(s')}{\left(\sum \phi_t(s')\right)^2}$$

$$= \frac{\phi_t(s)u_t(s)\sum \phi_t(s') - \phi_t(s)\sum \phi_t(s')u_t(s')}{\left(\sum \phi_t(s')\right)^2}$$

$$= \frac{\phi_t(s)u_t(s) - \phi_t(s)\overline{u}_t}{\sum \phi_t(s')} = \theta_t(s)\left(u_t(s) - \overline{u}_t\right)$$

Replicator Dynamics:

Proposition 1: Every NE is a steady state of the replicator dynamic.

Reason: All strategies in a NE have the same payoff.

Proposition 2: Any pure action is a steady state.

Then, we need to look at stability.

Review of Dynamical Systems:

Basic definitions:

State: $\theta_t(s) \in (0, 1)$, $s \in S$.

State vector: θ_t

Law of Motion: $\overset{\bullet}{\theta}_t = f(\theta_t)$

Flow: $F_t(\theta_0)$.

Definition 1: $\widehat{\theta}$ is a steady state if $F_t\left(\widehat{\theta}\right) = \widehat{\theta}$, for all t.

Definition 2: A steady state $\widehat{\theta}$ is stable if for all neighborhood U of $\widehat{\theta}$, there exists a neighborhood U_1 of $\widehat{\theta}$ such that if $\theta_0 \in U_1$, $F_t(\theta_0) \in U$ for all t.

Definition 3: A steady state $\widehat{\theta}$ is asymptotically stable if it is stable and in addition if $\theta_0 \in U_1$, then $\underset{t \to \infty}{lim} F_t(\theta_0) = \widehat{\theta}$.

Definition 4: A basin of attraction of a steady state $\widehat{\theta}$ is the set of points θ_0 such that $\underset{t \to \infty}{lim} F_t(\theta_0) = \widehat{\theta}$.

Replicator Dynamics:

Proposition 1: A stable steady state of the Replicator Dynamics is a Nash equilibrium of the stage game.

Proof: Assume not. Then, there exists s in the support of $\widehat{\theta}$ and $s' \in S$ such that $U\left(s', \widehat{\theta}\right) > U\left(s, \widehat{\theta}\right)$. By continuity of payoffs, for all θ' close to $\widehat{\theta}$ we have that $U\left(s', \theta'\right) > U\left(s, \theta'\right)$. But then $\theta_t(s) \rightarrow 0$.■

The converse is not necessarily true.

Example

	A	B
A	0, 0	0, 0
B	0, 0	1, 1

Proposition 2: An asymptotically stable steady state of the Replicator Dynamics is a trembling-hand-perfect and isolated Nash equilibrium of the stage game.

Replicator Dynamics and ESS:

Proposition 3: Every ESS is an asymptotically stabe SS of the Replicator Dynamics.

Reason: If s is ESS, $U\left(s, \varepsilon s' + (1-\varepsilon)s\right) > U\left(s', \varepsilon s' + (1-\varepsilon)\right.$
and s will grow faster.

Long Run Stability:

Kandori, M., Mailath, G. and Rob, R. (1993). "Learning, Mutation and Long Run Equilibria in Games," *Econometrica*, 61.

Study dynamics with persistent randomness.

Consider:

2x2 coordination game with A risk dominant.

N players.

Best response dynamic:

state θ is the number of As.

N large enough so that exists N^* such that $\alpha^* N \leq N^* \leq \frac{1}{2}N$.

$$\theta_{t+1} = \begin{cases} N \text{ if } \theta_{t+1} \geq N^* \\ 0 \text{ otherwise} \end{cases}$$

Two stable and ESS steady states:

$\theta = N$ and $\theta = 0$.

Long Run Stability:

Adding persistent mutations will help us eliminate one of the steady states!

At each point in time each player can mutate with probability ε.

What matters is in which basin of attraction the system is in.

$q_{AB} = P\left(\theta_{t+1} \in D_A / \theta_t \in D_B\right)$

Markov process, then find stationary distribution:

$$\begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix} = \begin{bmatrix} 1 - q_{BA} & q_{AB} \\ q_{BA} & 1 - q_{AB} \end{bmatrix} \begin{bmatrix} \varphi_A \\ \varphi_B \end{bmatrix}$$

$\frac{\varphi_A}{\varphi_B} = \frac{q_{AB}}{q_{BA}}$

Long Run Stability:

For BR dynamic with mutations ε:

$$\frac{\varphi_A^\varepsilon}{\varphi_B^\varepsilon} = \frac{q_{AB}^\varepsilon}{q_{BA}^\varepsilon} = \frac{\begin{pmatrix} N \\ N^* \end{pmatrix} \varepsilon^{N^*}(1-\varepsilon)^{N-N^*}}{\begin{pmatrix} N \\ N^* \end{pmatrix} \varepsilon^{N-N^*}(1-\varepsilon)^{N^*}}$$

$$\frac{\varphi_A^\varepsilon}{\varphi_B^\varepsilon} = \frac{(1-\varepsilon)^{N-2N^*}}{\varepsilon^{N-2N^*}}$$

Since $2N^* \leq N$: $\lim_{\varepsilon \to 0} \frac{\varphi_B^\varepsilon}{\varphi_A^\varepsilon} = 0$

Risk dominant survives in the long run.

Long Run Stability:

Step 1: Specify a state space θ.

Step 2: Specify an "intentional" dynamic: replicator, best response, Darwinian that may have many steady states.

Step 3: Introduce noise that makes the system ergodic (unique φ^{ε}).

Step 4: Find $\lim_{\varepsilon \rightarrow 0} \varphi^{\varepsilon} = \varphi^{*}$ (long run stable equilibrium).