Unpublished appendix for "Identification Issues in Forward-Looking Models Estimated by GMM, with an Application to the Phillips Curve"

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A.8 Under-identification test of section 3.1.1

Estimating the regression (8) by OLS, with $p = q = 4$ yields $R^2 = 0.78$, $\hat{\sigma}_v = 0.1$. Residual diagnostics, p-values in square brackets: AR 1-5 test: $F(5, 138) = 0.85$ [0.52], ARCH 1-4 test: $F(4, 135) = 0.38$ [0.82], Normality test: $Chi^2(2) = 0.54$ [0.76], White's heteroscedasticity test (no cross terms): $F(16, 126) = 1.26$ [0.23], with cross terms: $F(44, 98) = 1.34$ [0.12], RESET test: $F(1, 142) = 0.75$[0.39].

Testing under-identification hypothesis (10): Under the alternative, the model has 9 estimated parameters (including a constant), and there are 5 exclusion restrictions under H_0. The test statistic is $F(5, 143) = 0.58$ with the degrees of freedom given in the parentheses, and a p-value of 0.71.

Robustness: changing the number of lags under the alternative yields

p,q	F-stat	d.f.	p-value
5	0.54	7,141	0.81
6	0.66	9,139	0.75
7	0.67	11,137	0.76
8	0.62	13,135	0.84

AR(1): An F-test of the hypothesis that $\rho_i = \varphi_j = 0$ for $i > 1$, $j > 0$ in (8) against the alternative that at least one of those coefficients is nonzero yields $F(7, 143) = 0.60$ with a p-value of 0.75.

A.9 The rk-statistic

The rk-statistic is denoted rk(r), where r is the hypothesized rank of the $K \times 2$ matrix Π. Under H_0, rk(r) converges to a chi-square distribution with $(K - r)(2 - r)$ degrees of freedom (Kleibergen and Paap 2003, Theorem 1). In this application, $\mathrm{rk}(1) = T^{-1}\widehat{\boldsymbol{\lambda}}_1'\Omega_1\widehat{\boldsymbol{\lambda}}_1$, where $\widehat{\boldsymbol{\lambda}}_1$ depends on the smallest singular value of $\widehat{\Pi}$, the unrestricted OLS estimator of Π in the first-stage regression (14), while Ω_1 must be estimated using a HAC estimator of the variance of $Z_t V_t'$. The value reported in the text, rk(1) = 29.13, uses a Newey and West (1987) HAC estimator with lag truncation parameter set to 4. When no HAC estimator is used, the statistic reduces to the canonical correlations test of Anderson (1951), which is 20.52 [0.000] (p-value in brackets). When the Newey and West (1987) HAC is used in conjunction with prewhitening, à la Andrews and Monahan (1992), rk(1) becomes 31.05 [0.000]. When the lag truncation is varied from 1 to 12, the rk-statistic doesn't change much (from 27 to 31), and the p-value remains firmly at 0 to three decimal places.

A.10 Autocorrelated structural errors in the GG model

Consider the possibility that structural error in the baseline model (4) is autocorrelated, e.g., $\epsilon_t = \phi\epsilon_{t-1} + \varepsilon_t$, where ε_t is an innovation w.r.t. $\mathcal{F}_{t-1}$. Assuming s_t follows (11), for simplicity, the forward solution to the model is given by equation (12) with $\alpha_\epsilon = 1/(1-\gamma_f(\phi+\delta_1))$. From this, it follows that $\epsilon_t = \phi(\pi_{t-1} - \delta_1\pi_{t-2} - \alpha_0 s_{t-1} - \alpha_1 s_{t-2})(1-\gamma_f(\phi+\delta_1)) + \varepsilon_t$. Hence, the residual in the GMM regression (5) becomes

$$e_t = \phi(\pi_{t-1} - \delta_1\pi_{t-2} - \alpha_0 s_{t-1} - \alpha_1 s_{t-2})(1-\gamma_f(\phi+\delta_1)) + \varepsilon_t - \gamma_f\eta_{t+1} \tag{28}$$

Thus, s_{t-1}, s_{t-2} and π_{t-2} can be seen as variables incorrectly omitted from the estimated equation (5). In passing, note also that no lags of π_t and s_t would be valid instruments for the GMM regression (5), since they correlate with the residual.

A.11 Derivation of the coefficients of the forward solution (9)

The backward solution (21) is

$$\underbrace{\left[1+\alpha_v\varphi(L) - \gamma_f^{-1}\gamma(L)L\right]}_{a(L)}\pi_t \quad = \quad \underbrace{\left[\alpha_v\rho(L) - \gamma_f^{-1}\lambda(L)L\right]}_{b(L)}s_t + \underbrace{(\alpha_\epsilon - \gamma_f^{-1}L)}_{c(L)}\epsilon_t + \zeta_t.$$

The lag polynomials $a(L)$ and $b(L)$ are given by: coefficients of $a(L) = 1 - \sum_{i=1}^{\kappa_\pi+1} a_i L^i$, and $b(L) = \sum_{j=0}^{\kappa_s+1} b_j L^j$, $\kappa_s = \max(p-1,n)$ are given by

$$\begin{aligned}
a(L) &= 1+\alpha_v\varphi(L) - \gamma_f^{-1}\gamma(L)L \\
&= 1+\alpha_v\sum_{i=1}^{q}\varphi_i L^i - \gamma_f^{-1}\left(1-\sum_{i=1}^{m}\gamma_i L^i\right)L \\
&= 1+\left(\alpha_v\varphi_1 - \frac{1}{\gamma_f}\right)L + \sum_{i=2}^{\kappa_\pi+1}\left(\alpha_v\varphi_i + \frac{\gamma_{i-1}}{\gamma_f}\right)L^i, \quad \kappa_\pi = \max(q-1,m) && (29)\\
b(L) &= \alpha_v\rho(L) - \gamma_f^{-1}\lambda(L)L \\
&= \alpha_v\left(1-\sum_{j=1}^{p}\rho_j L^j\right) - \gamma_f^{-1}\sum_{j=0}^{n}\lambda_j L^{j+1} \\
&= \alpha_v - \sum_{j=1}^{\kappa_s+1}\left(\alpha_v\rho_j + \frac{\lambda_{j-1}}{\gamma_f}\right)L^j, \quad \kappa_s = \max(p-1,n). && (30)
\end{aligned}$$

Using the expansions

$$\begin{aligned}
\left(1-\frac{1}{z_0}L\right)\left(1-\sum_{i=1}^{\kappa_\pi}\delta_i L^i\right) &= 1-\left(\delta_1+\frac{1}{z_0}\right)L - \sum_{i=2}^{\kappa_\pi}\left(\delta_i - \frac{\delta_{i-1}}{z_0}\right)L^i + \frac{\delta_{\kappa_\pi}}{z_0}L^{\kappa_\pi+1} \quad \text{and}\\
\left(1-\frac{1}{z_0}L\right)\sum_{j=0}^{\kappa_s}\alpha_j L^j &= \alpha_0 + \sum_{j=0}^{\kappa_s}\left(\alpha_j - \frac{\alpha_{j-1}}{z_0}\right)L^j - \frac{\alpha_{\kappa_s}}{z_0}L^{\kappa_s+1}
\end{aligned}$$

together with (29) and (30), the identities $a(L) \equiv (1-L/z_0)\delta(L)$ and $b(L) \equiv (1-L/z_0)\alpha(L)$ can be written as

$$\begin{aligned}
1+\left(\alpha_v\varphi_1 - \frac{1}{\gamma_f}\right)L + \sum_{i=2}^{\kappa_\pi+1}\left(\alpha_v\varphi_i + \frac{\gamma_{i-1}}{\gamma_f}\right)L^i &\equiv 1-\left(\delta_1+\frac{1}{z_0}\right)L - \sum_{i=2}^{\kappa_\pi}\left(\delta_i - \frac{\delta_{i-1}}{z_0}\right)L^i + \frac{\delta_{\kappa_\pi}}{z_0}L^{\kappa_\pi+1}\\
\alpha_v - \sum_{j=1}^{\kappa_s+1}\left(\alpha_v\rho_j + \frac{\lambda_{j-1}}{\gamma_f}\right)L^j &\equiv \alpha_0 + \sum_{j=0}^{\kappa_s}\left(\alpha_j - \frac{\alpha_{j-1}}{z_0}\right)L^j - \frac{\alpha_{\kappa_s}}{z_0}L^{\kappa_s+1}.
\end{aligned}$$

Matching coefficients on both yields the equations

$$\begin{array}{rcl} 1/\gamma_f - \alpha_v \varphi_1 & = & \delta_1 + 1/z_0 \\ -\gamma_{i-1}/\gamma_f - \alpha_v \varphi_i & = & \delta_i - \delta_{i-1}/z_0, \quad i = 2, \ldots, \kappa_\pi \\ \alpha_v & = & \alpha_0 \\ -\alpha_v \rho_j - \lambda_{j-1}/\gamma_f & = & \alpha_j - \alpha_{j-1}/z_0, \quad j = 1, \ldots, \kappa_s. \end{array} \tag{31}$$

Re-arranging the above yields the recursive formulae given in (23).

A.12 Solution of the GG model

When π_t does not Granger-cause s_t, that is $\varphi(L) = 0$ in (8), the solution of (4) depends on the roots of the polynomial $1 - z/\gamma_f - z^2\gamma_b/\gamma_f = 0$. This follows from (19) by imposing the restriction $\varphi(z) = 0$ and substituting $\gamma(z) = 1 - \gamma_b z$. The two roots are $z_0 = (1 - \sqrt{\Delta})/(2\gamma_b)$ and $z_1 = (1 + \sqrt{\Delta})/(2\gamma_b)$, where $\Delta = 1 - 4\gamma_f\gamma_b$.

The underlying economic theory in GG links the parameters $(\lambda, \gamma_f, \gamma_b)$ in (4) to three 'deep' parameters measuring the discount factor, the degree of price inertia and the fraction of backward-looking agents in the economy (GG Equation 25). The fact that those deep parameters lie between 0 and 1 imposes restrictions on the admissible range of $(\lambda, \gamma_f, \gamma_b)$, namely $\lambda, \gamma_f, \gamma_b \geq 0$ and $\gamma_f + \gamma_b \leq 1$.

These restrictions imply that $1 \geq \Delta \geq (1 - 2\gamma_b)^2 \geq 0$, so the roots are positive real and distinct. Also, $z_0 - 1 = (1 - 2\gamma_b - \sqrt{\Delta})/(2\gamma_b) \leq 1 - 2\gamma_b - \sqrt{(1 - 2\gamma_b)^2} \leq 0$ and similarly, $z_1 - 1 \geq 0$, showing that $z_0 \leq 1$ and $z_1 \geq 1$, and hence, a solution exists. Both inequalities are strict when $\gamma_f + \gamma_b < 1$, implying that z_0 is explosive, and the model has a unique solution.

A.13 Derivation of equation (13)

Lead (12) one period, and substitute successively for π_t, s_{t+1} and s_t to get

$$\begin{array}{rcl} \pi_{t+1} & = & \alpha_0 s_{t+1} + \alpha_1 s_t + \delta_1 \pi_t + \alpha_\epsilon \epsilon_{t+1} \\ & = & \alpha_0(\rho_1 s_t + \rho_2 s_{t-1} + v_{t+1}) + \alpha_1 s_t + \delta_1(\alpha_0 s_t + \alpha_1 s_{t-1} + \delta_1 \pi_{t-1} + \alpha_\epsilon \epsilon_t) + \alpha_\epsilon \epsilon_{t+1} \\ & = & (\alpha_0\rho_1 + \alpha_1 + \delta_1\alpha_0)(\rho_1 s_{t-1} + \rho_2 s_{t-2} + v_t) + (\alpha_0\rho_2 + \delta_1\alpha_1)s_{t-1} + \delta_1^2 \pi_{t-1} \\ & & + \delta_1 \alpha_\epsilon \epsilon_t + \alpha_0 v_{t+1} + \alpha_\epsilon \epsilon_{t+1} \end{array}$$

which yields equation (13).