PERFECT EQUILIBRIA OF A MODEL OF *N*-PERSON NONCOOPERATIVE BARGAINING

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April 1993

1. Introduction

This paper investigates the set of subgame perfect equilibria associated with the model of noncooperative bargaining proposed by Hart and Mas-Colell (1992). The game form studied is defined over the class of cooperative games in coalitional form. For games with transferable utilities (TU games) Hart and Mas-Colell (1992) show that their game has a unique *stationary* subgame perfect equilibrium, the expected payoff from which is the Shapley value vector of the underlying coalitional game.

A key feature of the game form proposed by Hart and Mas-Colell (1992) is that it admits the possibility of *partial breakdown*, that is, situations where only a subset of the players are parties to the final agreement. This results from the novel feature that if a proposing player's offer to the other players is not unanimously accepted, there is an exogenous probability, $(1-\rho)$, that the proposer will be removed from the game. The parameter ρ, denoting the probability that the game will continue with the original set of players, plays much the same role as a discount factor.

While the stationary equilibrium does not depend on the parameter ρ, our analysis reveals that the set of perfect equilibrium payoffs depends crucially on the value of ρ. In this paper we report three results. All pertain to TU games.

(*i*) For small values of ρ, there is a unique perfect equilibrium payoff, the Shapley value payoff vector (Theorem 3.1 below). Thus for small ρ, restricting attention to stationary perfect equilibria does not restrict the set of perfect equilibrium payoffs.

(*ii*) When ρ is close to 1, typically there are multiple perfect equilibrium payoffs. However, the set of perfect equilibrium payoffs is *not* the same as the set of individually rational payoffs, and thus an implication of this is that in general, there is no "folk theorem" like result for the model at hand. Theorem 3.2 below provides an exact characterization of the set of perfect equilibrium payoffs for large ρ.

(*iii*) For the special class of symmetric games, we are able to provide a characterization of the set of perfect equilibrium payoffs for all values of ρ (Theorem 4.1 below). This result generalizes Shaked's analysis of the *n*-person pure bargaining problem (see Osborne and Rubinstein (1990)).

We hope to study the set of perfect equilibria associated with NTU games in a subsequent paper.

2. Preliminaries

Let (N, v) be a TU game in coalitional form. $N = \{1, 2, ..., n\}$ is the set of players and $v : 2^N \to R$ is the characteristic function defining the game. As always, $v(\emptyset) = 0$. The game (N, v) is assumed to be 0-*monotonic*, that is, for all $S \subset N$, and $i \notin S$, $v(S \cup \{i\}) \geq v(S) + v(i)$. We also assume that the game has been 0-*normalized* so that for all i, $v(i) = 0$.

Consider any set S of at least two players. For every i in S, define $t_{i,S}$ as

$$t_{i,S} = \frac{1}{s(s-1)} \sum_{j \in S\_i} v(ij) \qquad \mathbf{1}$$

(where $s = | S |$). Denote by t_s the s-vector whose ith component is $t_{i,S}$.

For every $S \subseteq N$ that consists of at least two players, define the set

$$T_s = \{x \in R^S : x(S) \leq v(S) \text{ and } x \geq t_s\}.$$

We will also write Eff $T_s = \{x \in R^S : x(S) = v(S) \text{ and } x \geq t_s\}$.

As an example, consider the three player game where $v(12) =$

12, $v(13) = 6$, $v(23) = 0$ and $v(123) = 12$. For this game $t_N = (3, 2, 1)$ and Figure 1 depicts Eff T_N. The point $b(1) = (v(N)-t_{2,N}-t_{3,N}, t_{2,N}, t_{3,N}) = (9, 2, 1)$. The points $b(2)$ and $b(3)$ are defined similarly.

The *Shapley value payoff* for player i in the game (N, v) is defined as:

$$Sh_i(N,v) = \sum_{\{R \subseteq N : i \in R\}} \frac{(r-1)!(n-r)!}{n!} [v(R) - v(R\_i)] \qquad 2$$

where as usual, $r = | R |$. Let Sh denote the Shapley value payoff vector.

It is instructive to note that we can rewrite (2) as:

$$\begin{aligned} Sh_i(N,v) = & \sum_{\{R \subseteq N : i \in R, r>2\}} \frac{(r-1)!(n-r)!}{n!} [v(R) - v(R\_i)] \\ & + \sum_{\{R \subseteq N : i \in R, r=2\}} \frac{1}{n(n-1)} [v(R) - v(R\_i)] \end{aligned} \qquad 3$$

and the second term in (3) is exactly $t_{i,N}$. Thus $t_{i,N}$ is just the term in the sum describing the Shapley value that consists of the expected marginal contribution of player i to single player coalitions. Because of 0-monotonicity, $Sh \in$ Eff T_N (as in Figure 1).

Next, suppose that $| N | > 2$ and observe that by definition:

$$\begin{aligned} \sum_{k \neq i} t_{i,N\_k} &= \sum_{k \neq i} \frac{1}{(n-1)(n-2)} \sum_{j \neq i,k} v(ij) \\ &= \frac{1}{(n-1)} \sum_{j \neq i} v(ij). \end{aligned} \qquad 4$$

This results in the following useful recursion:

$$t_{i,N} = \frac{1}{n} \sum_{k \neq i} t_{i,N\_k} \,. \qquad 5$$

Let $G(\rho, N)$ denote the game proposed by Hart and Mas-Colell

(1992). This is defined recursively. First, suppose $S = \{i, j\}$. The game $G(\rho, S)$ is defined as follows. One of the players i or j is chosen, with probability 1/2, as the proposer. Call $G^i(\rho, S)$ the game that results after i has been chosen as the proposer. In $G^i(\rho, S)$ player i makes an offer x such that $x_i + x_j \leq v(ij)$. If j accepts the offer the game ends. Otherwise, (*a*) with probability (1-ρ), the game ends, and both players receive 0; and (*b*) with probability ρ, $G(\rho, S)$ is played again, that is, the proposer is chosen at random and so on. Now suppose that $S \subseteq N$, with $|S| > 2$. In $G(\rho, S)$ one of the players, say i, is chosen with probability $1/s$ as the proposer. In the subgame $G^i(\rho, S)$, i makes a proposal x such that $x(S) \leq v(S)$ and each player in $S \backslash i$ is asked to either accept or reject i's proposal. The responses are made in increasing order of the player indices. If all accept the offer the game ends. Otherwise, (*a*) with probability (1-ρ) player i is removed from the game and receives a payoff of 0 and the players in $S \backslash i$ play the game $G(\rho, S \backslash i)$; and (*b*) with probability ρ, $G(\rho, S)$ is played again, that is, a player from S is chosen at random to make a new offer and so on. We refer to Hart and Mas-Colell (1992) for a detailed description, the interpretation of the game form and its motivation.

We wish to alert the reader that our formulation of $G(\rho, N)$ departs from that of Hart and Mas-Colell (1992) in two respects.

First, we 0-normalize the game so that for all i, $v(i) = 0$ whereas the Hart and Mas-Colell (1992) normalization allows $v(i) \geq 0$. This, of course, is an inessential difference.

Second, we have assumed that a proposer that is removed from the game receives a payoff of $v(i)$ (which is 0 because of our normalization). Hart and Mas-Colell (1992) have argued that in many situations it may be appropriate to "punish" a proposer that is removed from the game more severely, with a payoff $d_i \leq v(i)$. And certainly, this is more general. We have chosen the simpler specification in which, upon removal, a proposer's payoff is exactly $v(i)$ as it allows a simpler statements of our results.

While this is a substantive departure from the original formulation, in Section 5 below we indicate how our results can be easily amended to allow proposers to be punished with d_i rather than $v(i)$.

$PE[G(\rho, N)]$ will denote the set of payoffs from perfect equilibria of $G(\rho, N)$.

As a preliminary step, consider the case when $| N | = 2$. Arguments analogous to those for Rubinstein's (1982) two-person alternating offer game imply that there is a unique perfect equilibrium payoff. Thus we state without proof:

PROPOSITION 2.1: *Suppose* $| N | = 2$. *Then* $PE[G(\rho, N)] = \{(t_{1,N}, t_{2,N})\} = \{Sh\}$.

3. Perfect Equilibria

We begin with a result that shows that for small values of ρ, the game $G(\rho, N)$ has a unique perfect equilibrium payoff: the Shapley value. That such a result is true when $\rho = 0$, and thus $G(\rho, N)$ consists of a finite number of stages, has already been observed by Mas-Colell (1988) and Hart and Mas-Colell (1992). The result below suggests that for the proposed game the Shapley value is an especially salient outcome when the risk of breakdown is high.

THEOREM 3.1: *If* $\rho < 1/(n\text{-}1)$ *then* Sh *is the unique perfect equilibrium payoff of* $G(\rho, N)$.

PROOF: We argue by induction on the number of players. For two player games Proposition 2.1 implies that for all $\rho < 1$, there is a unique payoff vector. Suppose that for all $(n\text{-}1)$ player games $G(\rho, N\backslash i)$ the unique perfect equilibrium payoff is $Sh(N\backslash i, v)$ whenever $\rho < 1/(n\text{-}2)$. Suppose that $\rho < 1/(n\text{-}1) < 1/(n\text{-}2)$.

From Hart and Mas-Colell (1992) we know that $\{Sh\} \subseteq PE[G(\rho, N)]$. For all i, let m_i (resp. M_i) denote player i's infimum (resp. supremum) payoff in a perfect equilibrium of $G(\rho, N)$. Without loss of generality relabel the players so that $(Sh_1 - m_1) = \max\{(Sh_j - m_j) : j \in N\}$.

Let $\varepsilon > 0$ and consider the offer x by player 1 where for all $i \neq 1$, $x_i = (1-\rho)Sh_i(N\backslash 1, v) + \rho[v(N) - \sum_{j\neq i}m_j] + \varepsilon$ and $x_1 = v(N) - \sum_{i\neq 1}x_i$.

First, notice that for all $i \neq 1$, $x_i > (1-\rho)Sh_i(N\backslash 1, v) + \rho M_i$. Thus the offer x is sure to be accepted. Consider player n. If players 2, 3, ..., n-1 have already accepted, player n will do strictly worse by rejecting x. Thus n will accept if all others have already accepted. Now consider player n-1. If players 2, 3, ... , n-2 have accepted, then by accepting player n-1 will trigger an acceptance by player n and this is better for n-1 than rejecting. Working backwards in this fashion establishes that all players will accept x.

Next, observe that

$$\begin{aligned}\sum_{i=2}^{n} x_i &= (1-\rho)\sum_{i=2}^{n} Sh_i(N\_1, v) + \rho\sum_{i=2}^{n}[v(N) - \sum_{j\neq i} m_j] + (n-1)\varepsilon \\ &= (1-\rho)\sum_{i=2}^{n} Sh_i(N\_1, v) + \rho\sum_{i=2}^{n}[v(N) - \sum_{j=1}^{n} m_j + m_i] + (n-1)\varepsilon \qquad 6 \\ &= (1-\rho)v(N\_1) + \rho(n-1)\sum_{j=1}^{n}[Sh_j(N,v) - m_j] + \rho\sum_{i=2}^{n} m_i + (n-1)\varepsilon\end{aligned}$$

and thus,

$$\begin{aligned}x_1 &= v(N) - (1-\rho)v(N\_1) - \rho(n-1)\sum_{j=1}^{n}(Sh_j - m_j) - \rho\sum_{i=2}^{n} m_i - (n-1)\varepsilon \\ &= (1-\rho)[v(N) - v(N\_1)] - \rho(n-1)\sum_{j=1}^{n}(Sh_j - m_j) + \rho v(N) - \rho\sum_{i=2}^{n} m_i - (n-1)\varepsilon \qquad 7 \\ &= (1-\rho)[v(N) - v(N\_1)] - \rho(n-2)\sum_{j=1}^{n}(Sh_j - m_j) + \rho m_1 - (n-1)\varepsilon\end{aligned}$$

where we write $Sh_j = Sh_j(N, v)$.

Consider the following strategy for player 1: in $G^1(\rho, N)$ make the offer x and in $G^i(\rho, N)$ reject all offers. We can then write:

$$m_1 \geq \frac{1}{n} x_1 + \frac{1}{n} \sum_{i=2}^{n} [(1-\rho) Sh_1(N\_i, v) + \rho m_1]$$
$$= \frac{1}{n} x_1 + (1-\rho) \frac{1}{n} \sum_{i=2}^{n} Sh_1(N\_i, v) + \rho \frac{n-1}{n} m_1 \qquad \textbf{8}$$

and substituting from (7) we obtain:

$$m_1 \geq (1-\rho) \left[\frac{1}{n} [v(N) - v(N\_1)] + \frac{1}{n} \sum_{i=2}^{n} Sh_1(N\_i, v) \right]$$
$$- \rho \frac{n-2}{n} \sum_{j=1}^{n} (Sh_j - m_j) + \rho m_1 - \frac{n-1}{n} \varepsilon \qquad \textbf{9}$$

Since the term in brackets is exactly Sh_1 we can rewrite (9) as

$$(1-\rho)(Sh_1 - m_1) - \frac{n-1}{n} \varepsilon \leq \rho (n-2) \frac{1}{n} \sum_{j=1}^{n} (Sh_j - m_j) \qquad \textbf{10}$$

And since (10) holds for all $\varepsilon > 0$, we can infer that

$$(Sh_1 - m_1) \leq \left[\frac{\rho}{1-\rho} (n-2) \right] \frac{1}{n} \sum_{j=1}^{n} (Sh_j - m_j) \qquad \textbf{11}$$

But since $\rho < 1/(n-1)$, $[\rho(n-2)/(1-\rho)] < 1$ and thus if it were the case $(Sh_1 - m_1) > 0$, we would have that

$$(Sh_1 - m_1) < \frac{1}{n} \sum_{j=1}^{n} (Sh_j - m_j) \qquad \textbf{12}$$

which is impossible since for all j, $(Sh_1 - m_1) \geq (Sh_j - m_j)$. Thus $(Sh_1 - m_1) = 0$ and as a result for all j, $(Sh_j - m_j) = 0$. This completes the proof. □

Our main result concerning the perfect equilibrium payoffs of the game $G(\rho, N)$ when ρ is large is the following.

THEOREM 3.2: *There exists a* $\rho^* < 1$ *such that for all* $\rho \geq \rho^*$, *the set of perfect equilibrium payoffs of* $G(\rho, N)$ *is* T_N.

The proof of Theorem 3.2 is incorporated in Lemmas 3.1 and 3.2 below and uses some ideas from Shaked's analysis of the three person pure bargaining problem.

LEMMA 3.1: *For all* $\rho \in (0,1)$, $PE[G(\rho, N)] \subseteq T_N$.

PROOF: We will argue by induction on the number of players. By Proposition 2.1 the statement is true for all two-player games. So, suppose that for all sets S of $(n$-1) players every perfect equilibrium gives each player i at least $t_{i,S}$. Let N consist of n players.

Let m_i be player i's infimum payoff in a PE of $G(\rho, N)$. Consider the following strategy for i: at any stage, in any subgame of the form $G^i(\rho, N)$, ask for $v(N)$; and in any other subgame, regardless of the history, reject all proposals. Suppose that player k is the proposer. If i follows the strategy given above, with probability (1-ρ), k will be removed from the game and in the subsequent game $G(\rho, N\backslash k)$, by the induction hypothesis i's payoff will be at least $t_{i,N\backslash k}$. As a result we have that:

$$m_i \geq \frac{1}{n}\rho m_i + \frac{1}{n}\sum_{k \in N\_i}[(1-\rho)t_{i,N\_k} + \rho m_i] \qquad \mathbf{13}$$

Using (5) we can rewrite (13) as:

$$m_i \geq \frac{1}{n}\rho m_i + (1-\rho)t_{i,N} + \frac{n-1}{n}\rho m_i . \qquad \mathbf{14}$$

Rearranging and dividing by (1-ρ) then yields that $m_i \geq t_{i,N}$. □

LEMMA 3.2: *There exists a* $\rho^* < 1$ *such that for all* $\rho \geq \rho^*$, $T_N \subseteq PE[G(\rho, N)]$.

PROOF: Again we argue by induction. That the statement is true for two-player games is implied by Proposition 2.1. So suppose that there exists a ρ^{**} such that for all sets S consisting of n-1 or fewer players $\rho \geq \rho^{**}$ implies that $T_S \subseteq PE[G(\rho, S)]$.

Let N consist of n players and let Eff $T_N = \{x \in R^N : x(N) = v(N)$ and $x \geq t_N\}$ be the set of efficient payoff vectors in T_N.

Case 1: For all two-player coalitions S, $v(S) = v(N)$.

In this case for all i, $t_{i,N} = Sh_i$ and thus $T_N = \{Sh\}$. Then from Hart and Mas-Colell (1992) we know immediately that $\{Sh\} = T_N \subseteq PE[G(\rho, N)]$.

Case 2: There exists a two-player coalition S such that $v(S) < v(N)$.

We proceed in two steps. We first argue that the extreme points of Eff T_N can be supported as perfect equilibrium payoffs. We then argue that *any* point in T_N can also be supported.

Define $b(k, N) = (t_{1,N}, t_{2,N}, \ldots , t_{k-1,N}, [v(N) - \sum_{j\neq k} t_{j,N}], t_{k+1,N}, \ldots , t_{n,N})$ as the point in T_N where all players except k get a payoff exactly equal to the lower bound established in Lemma 3.1. The points $\{b(k, N) : k \in N\}$ constitute the n extreme points of Eff T_N and we will construct perfect equilibria which result in these payoffs. Similarly, $b(k, S)$ will denote the s extreme points of Eff T_S. (See Figure 1 for an illustration.)

For each k consider the n offers $z^i(k)$, $i \in N$. It may be helpful to think of this as an offer made by i that "rewards" k.

(i) The offer $z^k(k)$ is defined by: for all $j \neq k$, $z^k_j(k) = [(1-\rho)t_{j,N\backslash k} + \rho t_{j,N}]$ and $z^k_k(k) = [v(N) - \sum_{j\neq k} z^k_j(k)]$.

(ii) For $i \neq k$, $z^i(k)$ is defined by: for all $j \neq i,k$, $z^i_j(k) = [(1-\rho)t_{j,N\backslash i} + \rho t_{j,N}]$; $z^i_i(k) = \rho t_{i,N}$; and $z^i_k(k) = [v(N) - \sum_{j\neq k} z^i_j(k)]$.

First, notice that for all $j \neq k$

$$\frac{1}{n}\sum_{i=1}^{n} z^i_j(k) = \frac{1}{n}\sum_{i\neq j}[(1-\rho)t_{j,N\_i} + \rho t_{j,N}] + \frac{1}{n}\rho t_{j,N}$$

$$= (1-\rho)[\frac{1}{n}\sum_{i\neq j} t_{j,N\_i}] + \rho t_{j,N} \qquad 15$$

$$= t_{j,N} .$$

where we have used (5) again, and thus

$$\frac{1}{n}\sum_{i=1}^{n} z^i(k) = b(k). \qquad 16$$

Second, observe that:

$$\rho b_k(k,N) \leq z^k_k(k) \qquad 17$$

To see this, write (17) as:

$$\rho[v(N) - \sum_{j\neq k} t_{j,N}] \leq v(N) - (1-\rho)\sum_{j\neq k} t_{j,N\_k} - \rho\sum_{j\neq k} t_{j,N} \qquad 18$$

which is equivalent to

$$\sum_{j\neq k} t_{j,N\_k} \leq v(N). \qquad 19$$

But (19) is true since

$$\sum_{j\neq k} t_{j,N\_k} \leq v(N\_k) \leq v(N) . \qquad 20$$

Third, since $\sum_{j\in N} t_{j,N} < v(N)$, we have that for all k, $t_{k,N} < b_k(k, N)$ and since for all $l \neq k$, $z^k_k(l) = \rho t_{k,N}$, (17) immediately implies that: for $l \neq k$,

$$z^k_k(l) < z^k_k(k). \qquad 21$$

Finally, we establish an important fact.

CLAIM: *There exists a* ρ' *such that for* $\rho \geq \rho'$ *for all i, k,*

$$\frac{1}{(n-1)}[v(N) - \rho\, t_{i,N}] \leq (1-\rho)\, b_k(k, N\_i) + \rho\, b_k(k, N) \qquad \textbf{22}$$

PROOF OF CLAIM: Observe that:

$$\begin{aligned}\sum_{j \neq k} t_{j,N} - \frac{1}{(n-1)}\, t_{i,N} &= \sum_{j \neq i,k} t_{j,N} + \frac{(n-2)}{(n-1)} t_{i,N} \\ &= \frac{1}{n(n-1)} \sum_{j \neq i,k} \sum_{l \neq j} v(jl) + \frac{(n-2)}{(n-1)} \frac{1}{n(n-1)} \sum_{l \neq i} v(il) \\ &< (n-2)\frac{1}{n(n-1)}(n-1)v(N) + \frac{(n-2)}{(n-1)} \frac{1}{n(n-1)}(n-1)v(N) \\ &= \frac{(n-2)}{(n-1)} v(N)\end{aligned} \qquad \textbf{23}$$

where again we have used the assumption that for at least one two-player coalition *S*, $v(S) < v(N)$ and that every two player coalition occurs in the sum in (23). But (23) can be rearranged as

$$\begin{aligned}\frac{1}{(n-1)}[v(N) - t_{i,N}] &< v(N) - \sum_{j \neq k} t_{j,N} \\ &= b_k(k, N).\end{aligned} \qquad \textbf{24}$$

Thus for ρ large enough, (22) follows immediately. This establishes the claim.

Let $\rho^* = \max\{\rho^{**}, \rho'\}$ and suppose that $\rho \geq \rho^*$.

Consider the following strategies in $G(\rho, N)$, presented in the language of automata (see, for instance, Osborne and Rubinstein (1990)). The actions of players in any subgame are determined by the value of a "state" variable *k* that can take on values 1, 2, ... , *n*. Intuitively, *k* should be thought of as the identity of the player to be "rewarded"; all other players, including *j*, are "punished". Once an offer has been made the state variable is revised, if necessary. The strategies are described in Table 1 below where the indices i, j and k are distinct.

State	k	
Game	$G^k(\rho, N)$	$G^i(\rho, N)$
i	Accept x if $x_i \geq z_i^k(k)$	Offer $z^i(k)$
j	Accept x if $x_j \geq z_j^k(k)$	Accept x if $x_j \geq z_j^i(k)$
k	Offer $z^k(k)$	Accept x if $x_k \geq [(1-\rho)b_k(k, N\backslash i) + \rho b_k(k, N)]$
Transitio n Rule	If, in state k, the proposing player h proposes x with $x_h >$ z_h $^h(k)$, then go to state l, where $l \neq h$ is the player with the lowest index for whom $x_l < [v(N) - z_h^h(k)]/(n-1)$. Otherwise, the state remains k.	
	If, in state k, the game $G(\rho, N\backslash j)$ is played ($j \neq k$) play an equilibrium of $G(\rho, N\backslash j)$ that yields a payoff of $b(k, N\backslash j)$. If, in state k, the game $G(\rho, N\backslash k)$ is played, play an equilibrium of $G(\rho, N\backslash k)$ that yields a payoff of $t_{N\backslash k}$.	

Table 1

Perfect Equilibrium Strategies

We now verify that the strategies specified in Table 1 form a perfect equilibrium.

First, suppose the state is k and the game being played is $G^i(\rho, N)$ where $(i \neq k)$. Consider player j's rule $(j \neq i, k)$ for accepting offers. If any player responding before j has already rejected x, j's response is irrelevant. If there has been no previous rejection, by rejecting, in the next period, player j's expected payoff will be exactly $[(1-\rho)t_{j,N\backslash k} + \rho t_{j,N}] = z_j^k(k)$. Thus acceptance of x if and only if $x_j \geq z_j^k(k)$ is a best response by player $j \neq k$ to the other players' strategies. Now consider player k's rule for making offers when the state is k. Suppose k proposes x with $x_k > z_k^k(k)$. Then there is a player l such that $x_l < [v(N) - \rho t_{k,N}]/(n-1)$ and assume without loss of generality that the state will change to l. Then by the claim above, player l will find it profitable to reject and according to the strategies outlined above, player k will then get a payoff of: $(1-\rho)0 + \rho t_{k,N} = z_k^k(l)$. But from (21) we have that $z_k^k(l) < z_k^k(k)$. Thus k cannot profitably deviate and offer an x such that $x_k > z_k^k(k)$. Suppose k makes an offer such that for all j, $x_j \geq [v(N) - z_k$

$^{k}(k)]/(n\text{-}1)$. Then the state will remain k. Acceptance of this offer cannot benefit player k. If it is rejected, then according to the strategies outlined above, player k will then get a payoff of: $(1\text{-}\rho)0 + \rho b_k(k, N) \leq z_k^k(k)$, from (17). Thus, given the strategies of the others, player k cannot profitably deviate in $G^k(\rho, N)$.

Second, suppose the state is k and the game being played is $G^i(\rho, N)$, and consider player j's rule ($j \neq i,k$) for accepting offers. By the same argument as in the previous paragraph, acceptance of x if and only if $x_j \geq z_j$

$^{i}(k)$ is a best response by player $j \neq i,k$ to the other players' strategies. Now consider player k. In state k, player k's payoff in the next period will be $[(1\text{-}\rho)b_k(k, N\backslash i) + \rho b_k(k, N)]$ and thus acceptance of x if and only if it is at least as large as this amount is a best response for player k. Finally, the argument that player i is at a best response is also the same as in the previous paragraph.

Thus we have argued that the extreme points of Eff T_N are perfect equilibrium payoffs.

Finally, suppose $y \in T_N$. Then for all k, and $l \neq k$, $b_k(l, N) = t_{k,N} \leq y_k \leq b_k(k, N)$. Then we can find n offers y^i such that:

(i) $(1/n)\sum_{i \in N} y^i = y$;

(ii) for all k and $l \neq k$, $z_k^i(l) \leq y_k^i \leq z_k^i(k)$.

Notice that (ii) implies that for all $i \neq k$, $y_k^i \geq (1\text{-}\rho)t_{k,N\backslash i} + \rho t_{k,N}$ and $y_k^k \geq \rho t_{k,N}$.

We can now construct an equilibrium of $G(\rho, N)$ that results in a payoff of y as follows. We do this by "appending" an initial state, 0, to the equilibria implicit in Table 1 above. The equilibria of Table 1 are used as "punishments" to enforce the equilibrium outcome y. We now describe the amendments/additions.

Initially set the state variable to be 0. In state 0, in $G^i(\rho, N)$ player i makes the offer y^i. Any player $j \neq i$ accepts an offer x if and only if $x_j \geq (1-\rho)t_{j,N\backslash i} + \rho t_{j,N}$. If, in state 0, the proposing player i proposes x with $x_i > y_i^i$, then go to state l, where $l \neq h$ is the player with the lowest index for whom $x_l < [y(N) - y_i^i]/(n-1)$. Otherwise, the state remains 0.

In any state $k \neq 0$, the strategies clearly constitute a perfect equilibrium. To verify that the strategies are best responses when the state is 0, notice that $y(N) \leq v(N)$ and from (*ii*), $y_i^i \geq \rho t_{i,N}$. Hence $[y(N) - y_i^i]/(n-1) \leq [v(N) - \rho t_{i,N}]/(n-1)$ and thus from (22) we have that for all $\rho > \rho^*$, $[y(N) - y_i^i]/(n-1) < (1-\rho)b_k(k, N\backslash i) + \rho b_k(k, N)$. The rest of the verification is the same as for a state $k \neq 0$ and we omit the details. □

4. Symmetric Games

The game (N, v) is *symmetric* if for all $R, S \subseteq N$, $|R| = |S|$ implies that $v(R) = v(S)$. Thus such games can be completely described by $(n-1)$ numbers $(v_2, v_3, \ldots, v_n)$ where v_s is the worth of coalitions of size s. The 0-monotonicity condition implies that for all s, $v_s \leq v_{s+1}$. Also, for all i, $t_{i,N} = v_2/n$ and $Sh_i = v_n/n$.

For symmetric games we are able to provide a characterization of $PE[G(\rho, N)]$ for all values of ρ. It should be noted that for symmetric games a player's expected contribution to coalitions of size $(s-1)$ or less is v_s/n.

THEOREM 4.1: *Suppose that* (N, v) *is a symmetric game.*

(*i*) *If* $\rho < 1/(n-1)$,

$$PE[G(\rho, N)] = \{x : \forall i,\ x_i = v_n/n\};$$

(*ii*) *If for some* $s = 2, 3, \ldots, n-1$, $(1/s) \leq \rho < 1/(s-1)$,

$$PE[G(\rho, N)] = \{x : x(N) \leq v_n \text{ and } \forall i,\ x_i \geq v_s/n\}.$$

The statement of Theorem 4.1 is depicted in Figure 2 for the case where $n = 5$, and $(v_2, v_3, v_4, v_5) = (5, 10, 20, 30)$. For all values of ρ, the broken line labelled *m* (resp. *M*) depicts a player's minimum (resp. maximum) payoff in a perfect equilibrium of $G(\rho, N)$.

PROOF OF THEOREM 1: Since for all *i*, $Sh_i = v_n/n$, (*i*) is, of course, the same as Theorem 3.1.
(*ii*) Suppose that for some $s = 2, 3, \ldots, n-1$, $(1/s) \leq \rho < 1/(s-1)$.

We first argue that a player's infimum payoff in a perfect equilibrium, $m \geq v_s/n$. Again, we argue by induction on the number of players. If $n = 3$, $s = 2$ is the only possibility and then Lemma 3.1 immediately implies that $m \geq t_{i,N} = v_2/3$. Now suppose that for all games with n-1 players, a player's infimum payoff is at least $v_s/(n-1)$. Consider the following strategy for player 1: in the game $G^1(\rho, N)$ ask for v_n and in all other subgames reject all offers. This results in

$$m \geq \frac{1}{n}\rho m + \frac{n-1}{n}\left[(1-\rho)\frac{v_s}{n-1} + \rho m\right] \qquad \mathbf{25}$$

Rearranging (25) immediately yields that $m \geq v_s/n$.

Next define the set $V_N(s) = \{x \in \mathrm{R}^n : x(N) \leq v_n \text{ and } \forall i,\ x_i \geq v_s/n\}$. We now argue that the extreme points of Eff $V_N(s)$ can be supported as perfect equilibrium payoffs. Again, we proceed by induction on the number of players. If $n = 3$, $s = 2$ is the only possibility. Now notice that provided ρ is large enough, the result follows from Theorem 3.2 since $V_N(2) = T_N$. As in (22), ρ has to be large enough so that any non-equilibrium offer by a proposer which might be profitable is rejected by at least one responder. Using the induction hypothesis, we can write:

$$\frac{1}{2}[v_3 - \rho\frac{v_2}{3}] \leq (1-\rho)\frac{v_2}{2} + \rho[v_3 - 2\frac{v_2}{3}] \quad \textbf{26}$$

which reduces to:

$$(\frac{1}{2} - \rho)\, v_3 \leq (\frac{1}{2} - \rho)\, v_2. \quad \textbf{27}$$

Since $\rho \geq 1/2$ and $v_2 \leq v_3$, (27) holds.

Suppose that in all (n-1) player games $G(\rho, N\backslash i)$, all points in $V_{N\backslash i}(s)$ can be supported as perfect equilibrium payoffs. Consider $G(\rho, N)$ and the extreme points of Eff $V_N(s)$ which are of the form $c(k, N) = (v_s/n, \ldots, v_s/n, [v_n - (n-1)v_s/n], v_s/n, \ldots, v_s/n)$ where all players except k have a payoff of exactly v_s/n.

We can construct an equilibrium that results in $c(k, N)$ exactly in the same manner as we constructed an equilibrium that resulted in $b(k, N)$ in the proof of Lemma 3.2. It is sufficient to use v_s/n wherever we used $t_{i,N}$ and $v_s/(n-1)$ wherever we used $t_{i,N\backslash j}$ in that construction. Once again we only need to confirm that a non-equilibrium offer that is profitable for a proposer will be rejected by at least one responder if that responder is rewarded in the resulting subgame. Thus we need to verify that:

$$\frac{1}{(n-1)}[v_n - \rho\frac{v_s}{n}] \leq (1-\rho)[v_{n-1} - (n-2)\frac{v_s}{(n-1)}] + \rho[v_n - (n-1)\frac{v_s}{n}] \quad \textbf{28}$$

which can be rearranged as:

$$[\frac{1}{(n-1)} - \rho]\, v_n \leq (1-\rho)\, v_{n-1} - \frac{(n-2)}{(n-1)}\, v_s \quad \textbf{29}$$

But since $\rho \geq 1/(n-1)$, and $v_s \leq v_{n-1} \leq v_n$, we can write:

$$[\frac{1}{(n-1)} - \rho]\, v_n \leq [\frac{1}{(n-1)} - \rho]\, v_s$$
$$= (1-\rho)\, v_s - \frac{(n-2)}{(n-1)} v_s \qquad \textbf{30}$$
$$\leq (1-\rho)\, v_{n-1} - \frac{(n-2)}{(n-1)} v_s$$

which confirms (29).

Thus the extreme points of $V_N(s)$ can be supported as perfect equilibria. The rest of the proof may be completed as in Lemma 3.2. □

5. Extensions

Let $H(\rho, N)$ be the same as $G(\rho, N)$ except that once a proposal is rejected, with probability $(1-\rho)$ the proposer's payoff is $d_i \leq v(i)$. The payoffs d_i, $i \in N$ are fixed and given exogenously. Except for differences in normalization this is exactly the game formulated by Hart and Mas-Colell (1992). In this section we briefly indicate how our results about $G(\rho, N)$ may be extended to the more general game form $H(\rho, N)$. We emphasize that we continue to normalize the game so that for all i, $v(i) = 0$ and thus for all i, $d_i \leq 0$.

Proposition 2.1 and Theorem 3.1 remain true for $H(\rho, N)$ without any change.

Theorem 3.2 needs to be reformulated as follows. First, define for all S that consists of at least two players and every $i \in S$,

$$u_{i,S} = t_{i,S} + \frac{(s-2)}{s} d_i \,. \qquad \textbf{(1')}$$

Denote by u_S the vector whose ith component is $u_{i,S}$ and define the set

$$U_S = \{x \in R^S : x(S) \leq v(S) \text{ and } x \geq u_S\}.$$

Notice that if $|\ N\ | = 2$, $u_{i,N} = t_{i,N}$.

Then, analogous to Theorem 3.2, we can establish:

THEOREM 3.2′: *There exists a* $\rho^* < 1$ *such that for all* $\rho \geq \rho^*$, *the set of perfect equilibrium payoffs of* $H(\rho, N)$ *is* U_N.

And analogous to Theorem 4.1, we can establish:

THEOREM 4.1′: *Suppose that* (N, v) *is a symmetric game and for all* i, $d_i = d$.

(*i*) *If* $\rho < 1/(n-1)$,

$$PE[H(\rho, N)] = \{x : \forall i,\ x_i = v_n/n\};$$

(*ii*) *If for some* $s = 2, 3, \ldots, n-1$, $(1/s) \leq \rho < 1/(s-1)$,

$$PE[H(\rho, N)] = \{x : x(N) \leq v_n \text{ and } \forall i,\ x_i \geq [v_s + (n-2)d]/n\}.$$

6. References

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