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Reinterpreting the Kernel*

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The reduced game relevant for the consistency of the prekernel is used to reinterpret this solution concept in a way that makes it independent of interpersonal utility comparisons. Based on this reinterpretation, a non-cooperative model is presented that yields in stationary equilibria the prekernel (kernel) for the class of monotonic transferable utility games. Next, we apply the same non-cooperative model to more general coalitional games. For pure bargaining problems, the model yields the Nash solution. For general non-transferable utility games, a natural extension of the prekernel arises. Thus, the prekernel is obtained as the set of payoffs where the relevant bargaining range between every pair of players is split in half. *Journal of Economic Literature* Classification Numbers: C71, C72, C78.

1. INTRODUCTION

Hart and Mas–Colell [11] study a non-cooperative bargaining model to describe the transactions that take place in an underlying coalitional game. The stationary equilibrium outcomes of the non-cooperative game are studied when it is applied to different classes of cooperative problems.[1] First, when the game is applied to a transferable utility (TU) coalitional form, the Shapley value is obtained. Second, when the underlying structure is a pure bargaining problem, the game yields the Nash solution. Third, Hart and Mas–Colell show that for a general non-transferable utility (NTU) game, the limit of the stationary equilibria of their model yields the consistent Shapley values (see Maschler and Owen [18, 19]).[2]

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[1] See Krishna and Serrano [13] for a characterization of the non-stationary perfect equilibria of the Hart and Mas-Colell game when the coalitional game is TU.

[2] This solution concept is different from the Harsanyi [10] or the λ-transfer Shapley [35] values.

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The purpose of the exercise performed by Hart and Mas–Colell is at least threefold: to provide a non-cooperative implementation of well-known solution concepts, to uncover interesting relationships among them, and to use the non-cooperative analysis in order to extend the cooperative theory to domains where matters are much less settled. The main feature that the Hart/Mas–Colell bargaining game captures is the average of marginal contributions of players to coalitions, by choosing the proposer at random and allowing him and his resources to leave the game if his proposal to the grand coalition is rejected. This feature characterizes the consistent values, which have Shapley value and Nash solution as particular cases.

This paper presents an alternative and complementary theory to that of Hart and Mas–Colell [11]. Carrying out a similar exercise with a different non-cooperative model, we are able to obtain the prekernel in the TU case, the Nash solution in pure bargaining problems and a natural NTU prekernel that extends Moldovanu's [23] bargained equilibria for assignment problems.[3] Our non-cooperative model exhibits pairwise meetings of agents (for example, like those in Gale [7], Gul [8], Rubinstein and Wolinsky [30]). After the rejection of a proposal, the responder may (with some probability) threaten to break the grand coalition and form a coalition of his own that does not contain the proposer. The proposer, who is chosen at random in the pair, must take account of this threat when making a proposal. Thus, in equilibrium, the available surplus (modified by these outside options) is split in half.

With its strong Rawlsian content, the prekernel (and its single-valued selection, the prenucleolus) stands as a counterpart to the average-based Shapley value within the class of cooperative games with side payments or TU games. It equalizes the claims of every player against any other. At a payoff vector that is not in the kernel some player i can, by joining a coalition that does not contain player j, increase his utility by more than what player j's utility could increase had he joined a coalition that does not contain player i. On these grounds, player i is entitled to ask j for a compensation. The kernel was introduced in Davis and Maschler [6]. It is always contained in the bargaining set.[4] One drawback of the kernel (as argued in

[3] Extending Rochfort [27] to NTU games, Moldovanu concentrates on the intersection of the core and the kernel and his approach is based on a dynamic system without a strategic model of bargaining behind (see Stearns [36] or Maschler, Owen and Peleg [20] along the same lines). The only other extensions of the kernel to NTU games are found in Kalai [12] and Billera and McLean [4]. These solution concepts are unrelated to the kernel of the present paper: it is difficult to interpret them since they are based on excess functions for each coalition that lack a clear economic content.

[4] See Maschler [17] for a survey of the kernel and its properties.

the literature) is its heavy dependence on interpersonal comparisons of utility, although this paper will question this assertion.

Based on the same principles as the kernel but without the requirement of individual rationality is the prekernel.[5] The prekernel was axiomatized by Peleg [26] for the class of TU games. In Peleg's characterization, consistency and converse consistency are the central axioms.[6] Using the Davis–Maschler reduced game, which is relevant for the consistency property of the prekernel, this paper reinterprets this solution concept in a way that does not require interpersonal comparisons of utility: prekernel payoffs are reinterpreted as those in which the bargaining range between any two players is split in half. In fact, this interpretation was suggested earlier: Maschler, Peleg and Shapley [22] advanced it for the payoff vectors in the intersection of the prekernel and the ε-cores. The ε-cores are "windows" through which larger sections of the prekernel can be seen as ε (the parameter that measures the cost of forming a coalition) increases.

This paper does not use the geometry of the core in such an instrumental manner.[7] In contrast, it formalizes the bargaining content of the prekernel by means of a non-cooperative model that yields strategically Maschler *et al.*'s geometric result. Hence, a non-cooperative view of set-valued kernels is given for the first time.[8]

Finally, the non-cooperative model is used to explore a generalization of the prekernel to games with non-transferable utility. When the underlying problem is one of pure bargaining, the non-cooperative model yields in the limit the Nash solution as its unique equilibrium payoff. If the underlying coalitional game is a general NTU game, an intuitive and natural set of payoffs is found as the extension of the prekernel to this class of games. The prekernel is thus characterized as the set of payoffs in which every pair of players splits in half the relevant bargaining range. This range results from the consideration of the players' outside options that consist of threatening the opponent with breaking the grand coalition and forming a coalition that does not contain the threatened player. Equivalently, for every pair of players, the pair's Nash product of utilities is critical taking into account the endogenous disagreement point just referred to.[9] Particular cases of this

[5] In this paper we shall study monotonic games. Over this class, prekernel and kernel coincide.

[6] See Thomson [37] for a full appraisal of consistency.

[7] However, some connections exist. The reduced game relevant for the consistency of the core and the prekernel is the same (see Peleg [26]). Serrano [33] implements the core with a mechanism based on this reduced game.

[8] Serrano [31, 32] implements the nucleolus of different classes of games where the kernel is a singleton.

[9] In fact, this solution concept was introduced by Maschler, Owen and Peleg [20] for non-convex pure bargaining problems under the name of Nash set. However, its connection with the kernel was not uncovered. See also Herrero [38].

are the TU prekernel for TU games and the Nash solution for pure bargaining problems. Therefore, the prekernel extends the idea of equal division of the surplus to settings where coalitions are meaningful. We hope to see applications of the kernel to industrial organization problems or to the theory of contracts, where the Nash solution is often used as a device that splits the surplus equally.

The paper is organized as follows. Section 2 reviews games in coalitional form. The prekernel and its reinterpretation are studied in Sections 3 and 4. Section 5 introduces the non-cooperative model, and its results applied to different classes of games can be found in Sections 6, 7 and 8. Section 9 concludes with some examples and a brief discussion.

2. GAMES IN COALITIONAL FORM

The model underlying our non-cooperative procedure is a game in coalitional form (N, V). The set $N = \{1, 2, ..., n\}$ is the finite set of players (or factors of production) and V is a function that assigns a non-empty subset of $\mathbb{R}^s$ to each subset S of N (where $s = |S|$). Each point in $V(S)$ should be interpreted as a feasible payoff vector for the coalition S, given its initial endowments and/or production possibility set. We shall make the following standard assumptions on the coalitional function V.

A1. For each coalition S the set $V(S)$ is closed, convex and comprehensive, i.e., $V(S) - \mathbb{R}^s_+ \subset V(S)$. Moreover, $V(S) \cap \mathbb{R}^s_+$ is non-empty and bounded.

A2. 0-normalization: For all $i \in N$, $0 = \max t$ such that $t \in V(\{i\})$. This is just a normalization of the players' utility functions so that the utility level from the individual initial endowments and/or technology is 0.

A3. Monotonicity: $V(S) \times \{\mathbf{0}^{T \setminus S}\} \subset V(T)$ whenever $S \subset T$, i.e., if one completes a vector in $V(S)$ with zeros for the coordinates in $T \setminus S$, then one obtains a vector in $V(T)$. Cooperation is always worthwhile.

A4. The boundary of $V(N)$, denoted by $\partial V(N)$, is representable by a continuous function, i.e., there exists a continuous function $g \colon \mathbb{R}^n \to \mathbb{R}$ such that $\partial V(N) \equiv \{\mathbf{x} \in \mathbb{R}^n \colon g(\mathbf{x}) = 0\}$. Furthermore, at each point of $\partial V(N) \cap \mathbb{R}^n_+$ the gradient of g is uniquely defined and positive in all its coordinates. Thus, by an application of the implicit function theorem, we will denote by g_i the explicit function that expresses player i's coordinate at a boundary point as a function of the other $n-1$ coordinates. On the other hand, if $\mathbf{x} \in \partial V(N)$ with $x_i < 0$ for $i \in T$, $\partial V(N)$ at $\mathbf{x}$ is parallel to the subspace $\mathbb{R}^T$. This assumption is made so that all relevant action occurs in the positive orthant.

Two classes of games in coalitional form that have received much attention in the literature are those of games with transferable utility (TU) and pure bargaining problems. In the TU case it is assumed that for each coalition S there exists a real number $v(S)$ such that the feasible set for coalition S is $V(S) = \{\mathbf{x} \in \mathbb{R}^s \text{ such that } \sum_{i \in S} x_i \leqslant v(S)\}$. That is, the feasible set is the lower half-space of a hyperplane whose gradient is the vector consisting of s ones. In pure bargaining, for every coalition S other than the grand coalition N, $V(S) \subset -\mathbb{R}^s_+$, that is, subcoalitions have no more power than individual players. The results of this paper cover these two classes of problems, as well as general non-transferable utility (NTU) games.

3. THE KERNEL OF A TU COALITIONAL GAME

Let (N, v) be a TU game in coalitional form. The number $v(S)$ is usually interpreted as the initial resources of S as a group.[10] We assume that $v(\emptyset) = 0$. Notice that the Pareto frontier $\partial V(N)$ of the game (N, v) is the set $\{\mathbf{x} \in \mathbb{R}^n, \sum_{i \in N} x_i = v(N)\}$.

Definition. The *surplus* of player i against j at the payoff vector $\mathbf{x}$ is:

$$s_{i,j}(\mathbf{x}) \equiv \max_S \left\{ v(S) - \sum_{k \in S} x_k \right\}, \text{ over } S \text{ such that } i \in S,\ j \notin S.$$ [11]

That is, $s_{i,j}(\mathbf{x})$ is the maximum increase in utility that player i could obtain without the cooperation of player j by departing from $\mathbf{x}$ provided that the other players are paid at this rate.

Definition. The *prekernel* of the game (N, v) is the set

$$\text{Prk}(N, v) \equiv \{\mathbf{x} \in \partial V(N) : \text{for all } i \neq j,\ s_{i,j}(\mathbf{x}) = s_{j,i}(\mathbf{x})\}.$$

Definition. The *kernel* of the game (N, v) is the set

$$K(N, v) \equiv \{\mathbf{x} \in [\partial V(N) \cap \mathbb{R}^n_+] \text{ such that if } s_{i,j}(\mathbf{x}) > s_{j,i}(\mathbf{x}),\ x_j = 0\}.$$

[10] Since utility is transferable, $v(S)$ represents the maximum amount of utility available to the coalition S given its initial endowments and/or technology. Suppose, for example, that players' utility functions were quasilinear and that one of the goods that the coalition has is the numeraire.

[11] The amount $v(S) - \sum_{k \in S} x_k$ is sometimes referred to as the excess of coalition S at the payoff vector $\mathbf{x}$.

The prekernel and the kernel are meant to capture some idea of "equilibrium" between every pair of players. In the case of the prekernel, its allocations are such that the maximum utility that player i could gain by departing from $\mathbf{x}$ without counting on player j's cooperation is exactly the amount of utility that player j could gain by doing a similar move without the participation of player i. The same is true for the kernel subject to individual rationality. Namely, it could be that player i obtains more surplus than player j by departing from $\mathbf{x}$. However, player i cannot obtain a compensation from player j on those grounds since j's utility is already at its "minimum."

Remark. For every TU game (N, v) with a non-empty set of individually rational allocations, the prekernel and the kernel are always non-empty.

Remark. Under Assumptions A2 and A3, Theorem 2.7 in Maschler, Peleg and Shapley [21] shows that prekernel and kernel coincide. Therefore, all prekernel payoffs are individually rational in this case: if $\mathbf{x} \in \mathrm{Prk}(N, v)$, $x_i \geqslant 0$ for all $i \in N$.

4. REINTERPRETING THE PREKERNEL OF A TU GAME

Two observations on the definition of the kernel immediately spring to mind. First, the moves taken by the players should be understood in virtual time (it may very well happen that players i and j need the cooperation of intersecting coalitions). Second, the solution concept is heavily based on interpersonal comparisons of utility ($s_{i,j}(\mathbf{x})$ is measured in player i's utility terms whereas $s_{j,i}(\mathbf{x})$ is measured in j's). Both features are somewhat unappealing. In this section, we suggest a way to overcome them.

The prekernel was axiomatized by Peleg [26]. In Peleg's characterization, the central axioms are consistency and converse consistency in the sense of Davis and Maschler.

Definition. The Davis-Maschler *reduced game* for a coalition S and a payoff vector $\mathbf{x}$ is (S, v_{xS}), defined as

$$v_{xS}(S) = v(N) - \sum_{k \in N \setminus S} x_k; \qquad v_{xS}(\varnothing) = 0; \qquad \text{and}$$

$$v_{xS}(T) = \max_{Q \subseteq N \setminus S} \left\{ v(T \cup Q) - \sum_{k \in Q} x_k \right\} \qquad \text{for all other} \quad T \subset S.$$

The following results are shown in Peleg [26]:

(I) The prekernel satisfies consistency or the reduced game property: For all TU games (N, v), if $\mathbf{x} \in \text{Prk}(N, v)$, $\mathbf{x}^S \in \text{Prk}(S, v_{xS})$ for all $S \subset N$, where we denote by $\mathbf{x}^S$ the projection of $\mathbf{x}$ to the coordinate subspace corresponding to the players in S.

(II) The prekernel satisfies converse consistency or the converse reduced game property: Let $\mathbf{x} \in \partial V(N)$. If for all S, $|S| = 2$, $\mathbf{x}^S \in \text{Prk}(S, v_{xS})$, then $\mathbf{x} \in \text{Prk}(N, v)$.

Inspired by the Davis–Maschler reduced game, we can reinterpret the coalitional function as follows: Suppose that each of the agents owns an indivisible productive asset (the assets are indexed by the same indices as the players) and an amount of a divisible good called money that for convenience is normalized to 0. The coalitional function can then be viewed as representing the joint returns of the different portfolios of assets.[12] If preferences are quasilinear and a player ends up possessing the portfolio of assets S after paying an amount α for them, his final utility is $v(S) - \alpha$. This is a good interpretation of what the subcoalitions do in the Davis–Maschler reduced game. According to this interpretation, a payoff vector $\mathbf{x}$ is just a vector of "utility-prices" at which the n productive resources can be treated.

Consider the equation $s_{i,j}(\mathbf{x}) = s_{j,i}(\mathbf{x})$. That is,

$$\max_{i \in S,\, j \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\} = \max_{j \in S,\, i \notin S} \left\{ v(S) - \sum_{k \in S} x_k \right\},$$

which can be rewritten as

$$v_{x\{i,j\}}(\{i\}) - x_i = v_{x\{i,j\}}(\{j\}) - x_j,$$

or

$$2x_i = x_i + x_j + v_{x\{i,j\}}(\{i\}) - v_{x\{i,j\}}(\{j\}),$$

or

$$x_i = (1/2)[v_{x\{i,j\}}(\{i\})] + (1/2)\left[v(N) - \sum_{k \in N \setminus \{i,j\}} x_k - v_{x\{i,j\}}(\{j\}) \right].$$

Suppose that the vector of asset prices $\mathbf{x}$ has been proposed and that players other than an arbitrary pair $\{i, j\}$ accept these prices. We can interpret the amount $v(N) - \sum_{k \in N \setminus \{i,j\}} x_k$ as a "bargaining range" between players i and j, and the amounts $v_{x\{i,j\}}(\{i\})$ and $v_{x\{i,j\}}(\{j\})$ as players i and j's "outside options," respectively. The prekernel payoffs arise then as

[12] This interpretation also appears in Gul [8] and Serrano [33].

follows. Suppose players i and j are playing a one-period bargaining game with random proposer where the probability of being the proposer is 1/2 for both. The pie to split in these negotiations is $v(N)-\sum_{k\in N\setminus\{i,j\}} x_k$ and the responder has as an outside option to purchase assets from the portfolio $N\setminus\{i,j\}$, that is, the outside options are $v_{x\{i,j\}}(\{i\})$ and $v_{x\{i,j\}}(\{j\})$ for i and j, respectively.[13] Assume the proposer must pay $\sum_{k\in N\setminus\{i,j\}} x_k$ to the players in $N\setminus\{i,j\}$ for their assets and the value of the outside option to the responder. As a consequence, the "bargaining range" modified by outside options is split in half between players i and j. Since i and j were arbitrary, a price vector $\mathbf{x}$ with this property for all pairs $\{i,j\}$ must be a prekernel payoff.

Notice how the equations of the prekernel (that relied on interpersonal comparisons of utility) have been transformed into expected utility indifference conditions for each player. Likewise, note that the randomness on the proposer/responder roles solves the problem of the simultaneous use of resources of intersecting coalitions.

5. A NON-COOPERATIVE MODEL

Time runs discretely from $-\infty$ to $+\infty$. In each period t the productive possibilities of the economy underlying the non-cooperative model are described by the coalitional game (N, V). Also at the beginning of each period t there is a "status quo" vector of utilities $\mathbf{x}(t)$ determined by history. In each period t, there are exactly the same n myopic agents in the economy corresponding to the holders of the n different productive factors in the set N. In each period t $n-2$ agents will supply inelastically their productive factors to the economy at the "prices" $\mathbf{x}(t)$ while the other two (say, $i(t)$ and $j(t)$) are matched and have the opportunity of changing the prices of their productive assets through bargaining. In these negotiations, agents $i(t)$ and $j(t)$ are myopic, in that they do not take into account the fact that they will participate in some future negotiations (for example, because they do not know anything about the random matching process). The bilateral bargaining game played every period will be a modification of the procedure introduced by Hart and Mas–Colell [11]. The outcome of this bilateral bargaining process will determine a new "status quo" $\mathbf{x}(t+1)$ for the next period. In period $t+1$, the random matching process selects agents $i(t+1)$ and $j(t+1)$ to bargain given the new "status quo" and so on. Finally, we shall assume that the random process that selects pairs of agents to bargain is such that every pair is selected with positive probability in the future that starts at any given point in time.

[13] This is the bargaining game of the next section where $\rho=0$.

Myopia imposes that an agent not take into account that he will be involved in future negotiations when sitting at the bargaining table in a given period. For this reason, it will not be necessary to define the payoffs to an agent in the entire model. In sum, the model describes a situation in which most of the time the agents in the system are supplying their factors at the prevailing prices, and only rarely (say, once a year) do they have the opportunity to renegotiate their payments.

Like Hart and Mas–Colell [11], our methodology will be to test the same non-cooperative model against different underlying classes of coalitional functions. This will be done in the next three sections.

In each period t the pair $\{i, j\}$ (we drop the time argument for notational convenience) bargain over the feasible set $V(N) \mid \mathbf{x}(t)^{N \setminus \{i, j\}}$ defined as:

$$V(N) \mid \mathbf{x}(t)^{N \setminus \{i, j\}} = \{\mathbf{y} \in V(N) : \text{for all } k \in N \setminus \{i, j\}, \ y_k = \mathbf{x}_k(t)\}.$$

The interpretation is that the traders absent in the negotiations are supplying inelastically their productive assets at the prices $\mathbf{x}(t)$ in period t, while the bargainers have an opportunity to demonstrate on the bargaining table that the remuneration to their inputs should change.

Description of the meeting in period t: When players i and j meet, they bargain according to the following procedure. In stage 0 each of them simultaneously announces pre-bargaining settlements (z_i, z_j) such that $(z_i, z_j, \mathbf{x}(t)^{N \setminus \{i, j\}}) \in V(N)$.[14] If both announce the same settlement (z_i^*, z_j^*), both traders leave the bargaining table receiving these new flow payoffs, and the "status quo" for period $t+1$ is $\mathbf{x}(t+1) = (z_i^*, z_j^*, \mathbf{x}(t)^{N \setminus \{i, j\}})$. Otherwise, the following negotiation procedure (γ) begins between i and j.

The negotiation procedure γ (see Fig. 1): The proposer, chosen at random with equal probability, will make an offer (y_i, y_j) such that $(y_i, y_j) \in V(N) \mid \mathbf{x}(t)^{N \setminus \{i, j\}}$. If the responder accepts the offer, negotiations end and the new "status quo" is $\mathbf{x}(t+1) = (y_i, y_j, \mathbf{x}(t)^{N \setminus \{i, j\}})$.

If the responder rejects, there is a probability $1 - \rho$ of breakdown of negotiations $(0 \leqslant \rho < 1)$, whereas these continue with probability ρ according to the same negotiation procedure γ (of course, stage 0 is not repeated again). In the event of breakdown, the responder is entitled to use the factors in $N \setminus \{i, j\}$ in order to undertake the most profitable activity from his or her point of view and become its residual claimant. That is,

[14] We could get rid of this pre-bargaining stage altogether. It is included to keep randomness away from the equilibrium path.

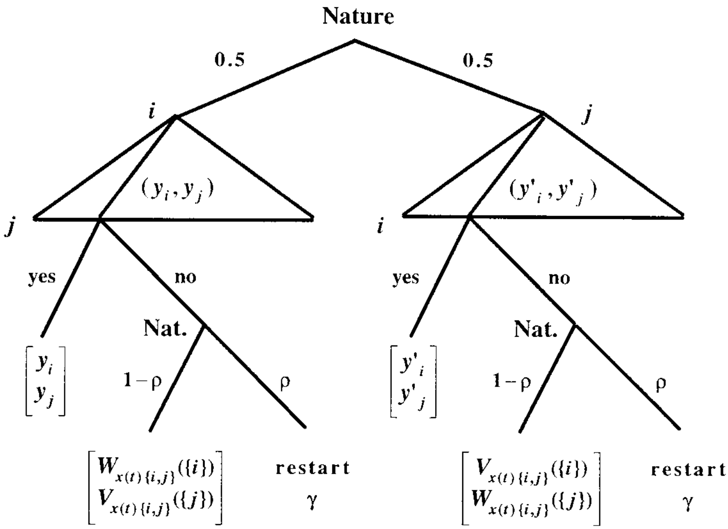

FIG. 1. The negotiation procedure γ (for a given status quo $x(t)$ and between players i and j).

assuming that the responder was j, player j can get the following generalization of the Davis–Maschler reduced game worth:

$$V_{x(t)\{i,j\}}(\{j\}) = \max_{Q \subset N \setminus \{i,j\}} \{u_j \text{ such that } (u_j, \mathbf{x}(t)^Q) \in V(\{j\} \cup Q)\}.$$

The proposer (say, player i), on the other hand, is forced to receive the maximum left compatible with feasibility (given our assumptions, this concept is well defined):

$$W_{x(t)\{i,j\}}(\{i\}) = g_i(V_{x(t)\{i,j\}}(\{j\}), \mathbf{x}(t)^{N \setminus \{i,j\}}).$$

Hence, the grand coalition always forms: if the responder has the threat $V_{x(t)\{i,j\}}(\{j\})$, the proposer is obliged to pay the responder that utility and get the remainder. Also notice that if all offers are rejected in this bargaining game, there is no need to specify the outcome since this event occurs with 0 probability (ρ^∞).

We assume that players have von Neumann–Morgenstern expected utilities. When there is an agreement between the bargainers or the breakdown in negotiations takes place, the players who bargain in period t leave the bargaining table with the flow payoffs attained. The bargainers in period $t+1$ inherit the status quo $\mathbf{x}(t+1)$. Notice that if there is disagreement in some stage of period t, the new status quo is actually a probability distribution over different vectors $\mathbf{x}(t+1)$.

Thus, in the economy described, payments to the factors of production are determined by the interplay of history and bilateral bargaining between all pairs of input holders. A holder of an input may change the payment to that factor in the bilateral bargaining rounds, in which the relative value of the outside options plays a crucial role. In these bargaining rounds, there is a friction captured by the probability of breakdown $1-\rho$. Most of the time we should think of ρ as being very close to 1 so that the probability of breakdown is small.

We shall introduce a simplifying assumption: there is a fixed protocol of matched pairs: for example, we can have $\{1, 2\}$ bargaining in period t, $\{2, 3\}$ in $t+1$, ..., $\{1, n\}$ in period $t-1+n(n-1)/2$.

The fixed protocol of pairwise meetings defines a cycle. Each cycle lasts exactly for $n(n-1)/2$ periods, i.e., the number of different pairs of players in the set N. Our solution concept will take into account this cycle.

Formally, the equilibrium notion incorporates the following elements:

(1) For every period t and every status quo $\mathbf{x}(t)$, bargainers i and j in period t employ stationary subgame perfect equilibrium strategies in their bargaining game. Their strategies can be made functions only of the status quo and of the identity of the opponent; and

(2) The status quo vectors $\mathbf{x}(t)$ describe a cycle of $n(n-1)/2$ periods of duration, where at most two components can change in each period. It is convenient to think of the cycle as being arranged in at most $n-1$ vectors of $\mathbb{R}^n_+ : \mathbf{x}^1, ..., \mathbf{x}^{n-1}$.

In bargaining games stationary SPE are not uncommon in the literature (we should point out that for the version of the result for TU games, one does not need to restrict attention to stationary SPE strategies). In the same spirit, the cyclicity requirement has the flavor of describing a stationary economy.

We shall be concerned with the limit as $\rho \to 1$ of these cyclical status quo vectors. In particular, we shall concentrate on *singleton limits*, i.e., those limits in which the cycle converges to a singleton: $\lim \mathbf{x}^1 = \cdots = \lim \mathbf{x}^{n-1}$. If we imagine $n-1$ agents of each type arranged in a circle and each of them bargains with his clockwise neighbor, we are asking that the payment to the productive resources of factor i not depend on the agent's position in the circle.

The model allows two additional interpretations. Suppose the economy consists of a continuum of agents with finite types (the n productive factor holders). Suppose further that the technological coefficients are fixed and equal to 1 in each productive process (that is, two agents of the same type should never participate in the same productive activity). One can think of a "market" in which pairs of selfish traders are selected from this continuum, are matched and bargain when it is their turn as representatives of their type's interests. Thus, the probability of being selected again to represent the class is 0. The outcome of the bargaining in the selected pair "sets up a new standard" and the changes in prices affect the whole continuum. Alternatively, consider an economy with overlapping generations of n agents that live for only one period. As soon as players $i(t)$ and $j(t)$ settle their differences through negotiations, the generation at period t is replaced by exact copies of themselves in such a way that the underlying economy continues to be described by the same coalitional function.

6. THE RESULT FOR TU GAMES

THEOREM 1. *Let (N, v) be a TU game satisfying A1–A3. The vector* $\mathbf{x}$ *is a singleton limit as $\rho \to 1$ of equilibrium payoffs of the non-cooperative model if and only if* $\mathbf{x} \in \mathrm{Prk}(N, v)$.

Proof. Notice first that for any status quo $\mathbf{x}$ if the coalitional function is TU, the outside option for player i when bargaining with j in period t is $V_{x\{i,j\}}(\{i\}) = v_{x\{i,j\}}(\{i\})$, while player i's payoff when player j uses his outside option is $W_{x\{i,j\}}(\{i\}) = v(N) - x_k - v_{x\{i,j\}}(\{j\})$.

We begin by analyzing the bilateral bargaining game that follows disagreement in the pre-bargaining settlement between two arbitrary players i and j in a period t where the status quo is an arbitrary $\mathbf{x}$.

LEMMA 1.1. *For any period t and for any status quo* $\mathbf{x}$, *there exists a unique subgame perfect equilibrium of the bilateral bargaining game between any pair of negotiating players i and j. Its expected payoff is*

$$z_i^* = (1/2)\left[v(N) - \sum_{k \in N\setminus\{i,j\}} x_k - v_{x\{i,j\}}(\{j\}) + v_{x\{i,j\}}(\{i\}) \right]$$

and

$$z_j^* = (1/2)\left[v(N) - \sum_{k \in N\setminus\{i,j\}} x_k - v_{x\{i,j\}}(\{i\}) + v_{x\{i,j\}}(\{j\}) \right].$$

Proof of Lemma 1.1. We first establish existence. Suppose the status quo is $\mathbf{x}$, and the bargainers in this period are i and j. We claim that the following pair of strategies constitutes a SPE of the modified Hart/Mas–Colell bilateral game:

(I) When acting as a proposer, player i offers

$$(z_i, z_j) = \left[v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - (1-\rho)\, v_{x\{i, j\}}(\{j\}) - \rho z_j^*,\right.$$
$$\left.(1-\rho)\, v_{x\{i, j\}}(\{j\}) + \rho z_j^* \right],$$

(II) When acting as a responder, player i accepts an offer (z_j, z_i) if and only if $z_i \geqslant (1-\rho)\, v_{x\{i, j\}}(\{i\}) + \rho z_i^*$,

(III) When acting as a proposer, player j offers

$$(z_j, z_i) = \left[v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - (1-\rho)\, v_{x\{i, j\}}(\{i\}) - \rho z_i^*,\right.$$
$$\left.(1-\rho)\, v_{x\{i, j\}}(\{i\}) + \rho z_i^* \right],$$

(IV) When acting as a responder, player j accepts an offer (z_i, z_j) if and only if $z_j \geqslant (1-\rho)\, v_{x\{i, j\}}(\{j\}) + \rho z_j^*$.

Note first that if these strategies are played, the outcome (z_i^*, z_j^*) results. Consider player i's payoff:

$$(1/2)\left[v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - (1-\rho)\, v_{x\{i, j\}}(\{j\}) - \rho z_j^* \right]$$
$$+ (1/2)[(1-\rho)\, v_{x\{i, j\}}(\{i\}) + \rho z_i^*] = z_i^*.$$

The computation is similar for player j's payoff. To check that the proposed strategies constitute a SPE, it should be clear that acting as a responder neither player has a profitable deviation by rejecting the equilibrium offer (which by construction leaves the responder indifferent). To show that the proposers are also playing a best response, notice that if they have a profitable deviation, it must entail a rejection (given the responder's strategy). Consider again player i as a proposer. If player j rejects i's offer, player i's continuation payoff is $(1-\rho)[v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - v_{x\{i, j\}}(\{j\})] + \rho z_i^*$. The reader can check that this expression equals $v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - (1-\rho)\, v_{x\{i, j\}}(\{j\}) - \rho z_j^*$, which is exactly the share that player i receives in equilibrium as a proposer. Therefore, the proposed strategy profile constitutes a SPE.

To show uniqueness, notice that the existence of the above SPE allows us to define m_i and m_j as player i's and j's infima over the set of SPE payoffs. Moreover, also from the existence of the above equilibrium, we know that $m_i \leqslant z_i^*$ and $m_j \leqslant z_j^*$.

Finally, consider the following strategy that player i can use: to reject every offer as a responder and to make an outrageous offer that player j will surely reject (notice that the offers are not required to satisfy individual rationality). Then, it must be true that $m_i \geqslant (1/2)\{(1-\rho)[v(N) - \sum_{k \in N \setminus \{i, j\}} x_k - v_{x\{i, j\}}(\{j\})] + \rho m_i\} + (1/2)\{(1-\rho)\, v_{x\{i, j\}}(\{i\}) + \rho m_i\}$, which yields $m_i \geqslant z_i^*$.

Clearly a similar strategy employed by player j shows that $m_j \geqslant z_j^*$. Therefore, $m_i = z_i^*$ and $m_j = z_j^*$, which concludes the proof of the Lemma. ∎

Therefore, in the pre-bargaining settlement, both players can announce the unique continuation payoffs (z_i^*, z_j^*).[15] Thus, it must be the case that $x_i = z_i^*(\{i, j\})$ for all i, j; where the last expression is the unique SPE payoff of the bargaining game between i and j. From the optimality of the SPE strategies, we have that every vector of n payments $\mathbf{x}$ so generated satisfies $\sum_{k \in N} x_k = v(N)$.

Taking into account the equations of Section 4 and the fact that the prekernel is always non-empty in the class of games considered establishes the existence of cyclical equilibrium payoffs for a fixed ρ. Finally, realize that the singleton limits of equilibrium payoffs must satisfy that for all $i \neq j$:

$$x_i = (1/2)\, v_{x\{i, j\}}(\{i\}) + (1/2) \left[v(N) - \sum_{k \in N \setminus \{i, j\}} \mathbf{x}_k - v_{x\{i, j\}}(\{j\}) \right],$$

which, again taking into account the equations of Section 4, completes the proof. ∎

Remark. Theorem 1 provides a partial implementation of different versions of the bargaining set (see Aumann and Maschler [3] and Mas–Colell [16]). Curiously, the origins of the kernel are related to the attempt to calculate portions of the bargaining set.

7. PURE BARGAINING PROBLEMS

The rest of the paper follows closely the agenda opened by Hart and Mas–Colell [11]. Specifically, we are interested in finding the outcomes that the non-cooperative model of Section 5 yields when the underlying

[15] There are other equilibria in the pre-bargaining phase, all with the same payoffs (whenever player i announces (z_i^*, z_j^*) and player j announces a different, less favorable for i, settlement).

coalitional function changes. In this section we study pure bargaining problems (see the end of Section 2 for a definition).

THEOREM 2. *Let (N, V) be a pure bargaining problem satisfying* A1–A4. *If there exists a singleton limit as $\rho \to 1$ of the equilibrium payoffs of the non-cooperative model, it is the Nash bargaining solution.*

Proof. Notice first that for every status quo $\mathbf{x}$, since (N, V) is a pure bargaining problem, we have that for all i and for all j, $V_{x\{i,j\}}(\{i\}) = 0$. For a fixed $\rho < 1$, we begin by studying the bargaining game between players i and j that follows a failure in the pre-bargaining settlement.

LEMMA 2.1. *Consider the bilateral bargaining game played by i and j when the status quo is $\mathbf{x}$ and the probability of breakdown is $1-\rho$. There exists a unique stationary subgame perfect equilibrium of this game. Its expected payoff satisfies the equations*

$$z_i^* = (1/2)\,\rho z_i^* + (1/2)\, g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})$$

and

$$z_j^* = (1/2)\,\rho z_j^* + (1/2)\, g_j(\rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}).$$

Proof of Lemma 2.1. If a stationary equilibrium of this game exists, it must satisfy the above equations (the responder is kept indifferent between accepting and rejecting, and the proposer takes the rest of the available surplus). Denote by $\alpha(\mathbf{z})$ the mapping whose components are the right hand sides of the two equations above. For the domain of vectors $\mathbf{z} = (z_i, z_j)$ take the following set:

$$A = \{[0, W_{x\{i,j\}}(\{i\})] \times [0, W_{x\{i,j\}}(\{j\})]\} \cap \{V(N) \mid \mathbf{x}^{N\setminus\{i,j\}}\}.$$

Notice that for all $\mathbf{x}^{N\setminus\{i,j\}}$, A is non-empty, compact and convex. Also, $\alpha(\mathbf{z})\colon A \to A$ is continuous given our assumptions. Therefore, by Brouwer's fixed point theorem, α has a fixed point (z_i^*, z_j^*). Notice that, when $\partial V(N)$ is not linear, the fixed point is interior to $V(N)$ since it is the average of two points of the frontier.

The strategies that support this stationary outcome are the following:

(I) As a proposer, player i offers $(g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}), \rho z_j^*)$;

(II) As a responder, player i accepts an offer (z_i, z_j) made by player j if and only if $z_i \geqslant \rho z_i^*$.

(III) As a proposer, player j offers $(\rho z_i^*, g_j(\rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}))$;

(IV) As a responder, player j accepts an offer (z_i, z_j) made by player i if and only if $z_j \geqslant \rho z_j^*$.

It is easy to see that the responders do not have an incentive to deviate because they are kept exactly indifferent between accepting and rejecting. The proposers are also playing a best response: consider player i. If he deviates and makes a different proposal, either it is accepted, which (given the strategy of player j) cannot be profitable; or it is rejected. In the latter case, player i's expected payoff from the deviation would be $(1-\rho)\, W_{x\{i,j\}}(\{i\}) + \rho z_i^* \leqslant g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})$, by convexity of $V(N)$. To see this, recall the definition of g_i and notice that it is a concave function of j's argument. Therefore

$$\begin{aligned} g_i((1-\rho)\, 0 + \rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}) &\geqslant (1-\rho)\, g_i(0, \mathbf{x}^{N\setminus\{i,j\}}) + \rho g_i(z_j^*, \mathbf{x}^{N\setminus\{i,j\}}) \\ &= (1-\rho)\, W_{x\{i,j\}}(\{i\}) + \rho g_i(z_j^*, \mathbf{x}^{N\setminus\{i,j\}}) \\ &\geqslant (1-\rho)\, W_{x\{i,j\}}(\{i\}) + \rho z_i^*. \end{aligned}$$

To prove uniqueness, rewrite the equations as

$$z_i^* = [1/(2-\rho)]\; g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})$$

and

$$z_j^* = [1/(2-\rho)]\; g_j(\rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}).$$

Therefore, all stationary equilibrium payoffs in the bilateral game must satisfy that $z_i^* g_j(\rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}) = g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})\, z_j^*$. Multiplying both sides by ρ, both points $(\rho z_i^*, g_j(\rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}))$ and $(g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}), \rho z_j^*)$, whose average is (z_i^*, z_j^*), lie on the same level curve of $y_i y_j$.

Thus suppose there were a second stationary equilibrium (z_i', z_j') as the average of the points $(\rho z_i', g_j(\rho z_i', \mathbf{x}^{N\setminus\{i,j\}}))$ and $(g_i(\rho z_j', \mathbf{x}^{N\setminus\{i,j\}}), \rho z_j')$. Assume without loss of generality that $|z_i'| > |z_i^*|$. Hence, $|g_i(\rho z_j', \mathbf{x}^{N\setminus\{i,j\}})| > |g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})|$. But then it would be impossible that the extreme points that generate (z_i', z_j') lie on the same hyperbola of $y_i y_j$, which is a contradiction.

This concludes the proof of the Lemma. ∎

By the last part of our assumption A4, for every pair $\{i, j\}$ the stationary SPE payoffs of their bargaining game have positive coordinates. As $\rho \to 1$ we consider the functions $(z_i^*(\rho), z_j^*(\rho))$ and we drop their dependence on ρ for notational simplicity. Notice that $(\rho z_i^*, g_j(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}))$ and $(g_i(\rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}), \rho z_j^*)$ converge to the same point. Since along the sequence, the two points lie on the same level curve of $y_i y_j$, the limit point, which is efficient, must lie on the highest of such level curves. That is, the limit point (n_i, n_j) must be the Nash bargaining solution of the bilateral problem: $n_i/n_j = (D_i g)_n/(D_j g)_n$, where Dg is the gradient of the frontier of $V(N) \mid \mathbf{x}^{N\setminus\{i,j\}}$.

Therefore, if there exists a cyclical equilibrium, it is straightforward to check that its singleton limit must be the Nash bargaining solution of the multilateral problem: that is, the only point $\mathbf{x}$ of $\partial V(N)$ in which all pairs $\{i, j\}$ are agreeing on the Nash solution of the induced bilateral problem $V(N) \mid \mathbf{x}^{N \setminus \{i, j\}}$.

The only thing left to prove is the existence of a cyclical equilibrium for a fixed ρ. This will be done in Theorem 3 for the more general case and therefore the proof is omitted here. ∎

Remark. A sufficient condition for the existence of a singleton limit is that the game (N, V) be generated by a "splitting a one-dimensional pie." The proof is based on the monotonicity of the Nash solution over this class and is left to the reader.

Remark. Theorem 2 provides a non-cooperative view of the Nash bargaining solution based on the property of converse consistency. In contrast, the extensive form in Krishna and Serrano [14] is inspired by the dual property of consistency. See Harsanyi [9] and Lensberg [15]. Also, the reader will have noticed the resemblance of the proof of convergence to the bilateral Nash solution with the result of Binmore, Rubinstein and Wolinsky [5].

8. NTU GAMES

The extension of the prekernel to NTU games proposed in this paper is given in the following definition. This definition comes out from the analysis of the same non-cooperative model. Recall our Assumption A4 about the coalitional function and, in particular, the representation of the frontier $\partial V(N) \cap \mathbb{R}^n_+$ by a function $g(\mathbf{x}) = 0$. We shall denote by $D_i g$ the partial derivative of g with respect to x_i.

DEFINITION. The prekernel of a NTU game is the set

$$\begin{aligned} \operatorname{Prk}(N, V) \equiv \{\mathbf{x} \in \partial V(N)\colon \text{for all } i \neq j,\ (D_i g)_x\, (V_{x\{i, j\}}(\{i\}) - x_i) \\ = (D_j g)_x\, V_{x\{i, j\}}(\{j\}) - x_j\}. \end{aligned}$$

That is, the surplus of player i against player j is the increase of utility (weighted by the marginal value of his utility at that point) that he could get without cooperating with j by hiring his most preferred coalition at the rate $\mathbf{x}$. To see how this definition can be made independent of interpersonal comparisons of utility, notice that the following is an equivalent definition (closer to Nash's bargaining theory):

$\text{Prk}(N, V) \equiv \{\mathbf{x} \in \partial V(N)$ such that for all $i \neq j$, (x_i, x_j) is a critical point of the bilateral Nash product $(y_i - V_{x\{i,j\}}(\{i\}))(y_j - V_{x\{i,j\}}(\{j\}))$ subject to $(y_i, y_j) \in \partial V(N) \mid \mathbf{x}^{N \setminus \{i,j\}}\}$.

Thus, in this set of efficient payoffs, every pair of players finds a critical value of the product of the differences between the utility obtained at **x** and the "disagreement" utility, where the latter is defined as the amount of utility that a player can achieve (without counting on the cooperation of the other) by choosing his most preferred coalition, while paying them at the rate **x**. This is the set of payoffs that the non-cooperative model yields: for each pair of players, the relevant bargaining range is split in half, as shown in the proof of Theorem 3. Notice that the Nash product for each pair is not necessarily maximized: even with monotonic games, it may happen that the "disagreement point" $(V_{x\{i,j\}}(\{i\}), V_{x\{i,j\}}(\{j\}))$ for a pair $\{i, j\}$ lies outside of the induced feasible set $V(N) \mid \mathbf{x}^{N \setminus \{i,j\}}$, in which case their Nash product could be minimized at **x**. Of course, this will never happen if the game (N, V) represents a pure bargaining problem, and only maximizers of the bilateral Nash products are picked up by the solution concept.

THEOREM 3. *Let (N, V) be a NTU game satisfying A1–A4. The singleton limit as $\rho \to 1$ of the equilibrium payoffs of the non-cooperative model is* $\text{Prk}(N, V)$.

Proof. Recall the definitions of $V_{x\{i,j\}}(\{i\})$ (player i's outside option against j) and $W_{x\{i,j\}}(\{i\})$ (player i's highest payoff when player j exercises his outside option).

For a fixed $\rho < 1$, we begin by studying the bargaining game between players i and j that follows a failure in the pre-bargaining settlement.

LEMMA 3.1. *Consider the bilateral bargaining game played by i and j when the status quo is* **x** *and the probability of breakdown is* $1 - \rho$. *Stationary subgame perfect equilibria of this game exist, all of which satisfy that their expected payoffs are given by the equations*

$$z_i^* = (1/2)[(1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*] \\ + (1/2)\, g_i((1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*, \mathbf{x}^{N \setminus \{i,j\}})$$

and

$$z_j^* = (1/2)[(1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*] \\ + (1/2)\, g_j((1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*, \mathbf{x}^{N \setminus \{i,j\}}).$$

Proof of Lemma 3.1. If a stationary equilibrium of the game exists, it must satisfy the above equations (the responder is kept indifferent between accepting and rejecting, and the proposer takes the rest of the available surplus). Denote by $\alpha(\mathbf{z})$ the mapping whose components are the right hand sides of the two equations above. For the domain of $\mathbf{z}=(z_i, z_j)$ take the following set:

$$\begin{aligned} A = \{ & [\min\{V_{x\{i,j\}}(\{i\}), W_{x\{i,j\}}(\{i\})\}, \\ & \quad \max\{V_{x\{i,j\}}(\{i\}), W_{x\{i,j\}}(\{i\})\}] \\ & \times [\min\{V_{x\{i,j\}}(\{j\}), W_{x\{i,j\}}(\{j\})\}, \\ & \quad \max\{V_{x\{i,j\}}(\{j\}), W_{x\{i,j\}}(\{j\})\}]\} \\ & \cap \{V(N) \mid \mathbf{x}^{N\setminus\{i,j\}}\}. \end{aligned}$$

Notice that for all $\mathbf{x}^{N\setminus\{i,j\}}$, A is non-empty, compact and convex. Also, $\alpha(\mathbf{z})\colon A \to A$ is continuous given our assumptions. Therefore, by Brouwer's fixed point theorem, α has a fixed point (z_i^*, z_j^*). Notice that, given the possible non-linearity of the Pareto frontier of $V(N)$, the fixed point may be interior to $V(N)$ since it is the average of two points of the frontier.

The strategies that support this stationary outcome are the following:

(I) As a proposer, player i offers $(g_i((1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}), (1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*)$;

(II) As a responder, player i accepts an offer (z_i, z_j) made by player j if and only if $z_i \geqslant (1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*$;

(III) As a proposer, player j offers $((1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*, g_j((1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}))$;

(IV) As a responder, player j accepts an offer (z_i, z_j) made by player i if and only if $z_j \geqslant (1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*$.

It is easy to see that the responders do not have an incentive to deviate because they are kept exactly indifferent between accepting and rejecting. The proposers are also playing a best response: consider player i. If he deviates and makes a different proposal, either it is accepted, which (given the strategy of player j) cannot be profitable; or it is rejected. In the latter case, player i's expected payoff from the deviation would be $(1-\rho)\, W_{x\{i,j\}}(\{i\}) + \rho z_i^* \leqslant g_i((1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})$, by convexity of $V(N)$. The argument is identical to the same step in the proof of Lemma 2.1 and we omit it here.

This concludes the proof of the Lemma. ∎

Notice that we can rewrite the equations that define the stationary equilibria as

$$\begin{aligned} z_i^* = {} & (1/2)\, V_{x\{i,j\}}(\{i\}) \\ & + (1/2)(g_i((1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}}) - \rho z_i^*)/(1-\rho) \end{aligned}$$

and

$$z_j^* = (1/2)\, V_{x\{i,j\}}(\{j\}) + (1/2)(g_j((1-\rho)\, V_{x\{i,j\}}(\{i\}) + \rho z_i^*, \mathbf{x}^{N\setminus\{i,j\}}) - \rho z_j^*)/(1-\rho).$$

As $\rho \to 1$ the sequences of outside option utilities $V_{x\{i,j\}}(\cdot)$ and of residual utilities g_i when the other player opts out have a limit since they live in appropriately defined compact sets. Then, as $\rho \to 1$ we have that in the limit, $(z_i^*, z_j^*) \in \partial V(N) \mid \mathbf{x}^{N\setminus\{i,j\}}$. Furthermore, using L'Hopital's rule (notice that $g_i((1-\rho)\, V_{x\{i,j\}}(\{j\}) + \rho z_j^*, \mathbf{x}^{N\setminus\{i,j\}})$ converges to ρz_i^*), we obtain that

$$\lim z_i^* = (1/2) \lim V_{x\{i,j\}}(\{i\}) + (1/2)[\lim (dg_i/dy_j)(\lim z_j^* - \lim V_{x\{i,j\}}(\{j\})) - \lim z_i^*]/(-1),$$

or

$$\lim [z_i^* - V_{x\{i,j\}}(\{i\})] = \lim [(\partial g/\partial y_j)/(\partial g/\partial y_i)] \lim [z_j^* - V_{x\{i,j\}}(\{j\})].$$

If a cyclical equilibrium exists, we shall have that all singleton limits of the equilibria must satisfy that for all $i \neq j$, the equation just written holds and therefore the singleton limit would be a point in Prk(N, V).

The only thing left to prove is the existence of a vector $\mathbf{x} \in \mathbb{R}^{n(n-1)}$ that corresponds to the payoffs of a cyclical equilibrium for a fixed $\rho < 1$. Consider the $n(n-1)$ equations of Lemma 3.1 when one takes into account all $n(n-1)/2$ possible pairs of players.

Fix $\mathbf{x} = \mathbf{0} \in \mathbb{R}^{n(n-1)}$ and consider the following interval for the payoff of the $n-1$ types in the cycle of an arbitrary player i:

$$A_i = [\min_{j \neq i} \{V_{0\{i,j\}}(\{i\}), W_{0\{i,j\}}(\{i\})\}, \max_{j \neq i} \{V_{0\{i,j\}}(\{i\}), W_{0\{i,j\}}(\{i\})\}].$$

Consider then the set $A' = \prod_{i \in N} (n-1)\, A_i \cap (n-1)\, V(N)$, where $(n-1)\, Z$ denotes the $(n-1)$-Cartesian product of an n-dimensional set of payoffs. $A' \subset \mathbb{R}^{n(n-1)}$ is non-empty, compact and convex. Define the following mapping: $\beta(\mathbf{x})$: $A' \to A'$ using the $n(n-1)$ equations of Lemma 3.1. The mapping β is continuous and by Brouwer's fixed point theorem it has a fixed point $\mathbf{x}^*$. ∎

Remark. Thus, the existence of a vector $\mathbf{x} \in \mathbb{R}^{n(n-1)}$ satisfying the equations of Lemma 3.1 is supported by standard arguments, given the assumptions we made in Section 2. This means that a cyclical equilibrium will exist. The same will not be true for the singleton limit of the equilibria, i.e., it is perfectly possible that, as $\rho \to 1$, a given sequence of cyclical equilibria not yield the same payoff to the $n-1$ "versions" of a player.

9. EXAMPLES AND DISCUSSION

The theory of the kernel encountered difficulties in its extension to general games where utility is not transferable. The main problem seemed to be the dependence of the solution on interpersonal comparisons of utility. This paper has proposed a natural extension of the prekernel by finding that it is the solution concept that prescribes that each pair of players must split in half the available surplus that exists in the bilateral problem once one takes into account the threat point in which each of them could take his most preferred coalition that excludes the threatened player. Not surprisingly, in pure bargaining this solution is found to coincide with Nash's [24, 25].

The reader may wonder whether the NTU prekernel found in this paper can be given an axiomatic support. Serrano and Shimomura [34] answer this question in the affirmative, by combining the axiom systems in Peleg [26] and Nash [24].

The following are some examples of NTU prekernels intended to show how it is computed and to clarify some of its general properties. In them, for simplicity, we shall not truncate the set $V(N)$.

EXAMPLE 1.[16] Let $N=\{1, 2, 3\}$. For every $i \in N$, $B(\{i\})=\{\mathbf{x} \in \mathbb{R}$ such that $\mathbf{x} \leqslant 0\}$; $V(\{1, 2\})$ is the comprehensive hull of $(1/2, 1/2)$; $V(\{1, 3\})$ and $V(\{2, 3\})$ are each of them the comprehensive hull of $(p, 1-p)$ where $p<1/2$; $V(N)$ is the comprehensive hull of the convex hull of the points $(1/2, 1/2, 0)$, $(p, 0, 1-p)$ and $(0, p, 1-p)$. Not to have boundary problems, assume that $\partial V(N)$ is the TU frontier where all three payoffs add up to 1. The reader can calculate that $\text{Prk}(N, V)=\{(1/2, 1/2, 0)\}$: given that player 3 receives 0, player 1 could threaten player 2 to hire 3, in which case player 1 would get a utility of p, but the same threat can be done by player 2 against 1. Also, player 1 could threaten player 3 to hire 2 and therefore get 1/2, whereas (given that player 2 is being paid 1/2), player 3's counterargument has a maximum utility of 0. This contrast with the Harsanyi value, the λ-transfer Shapley value or the consistent value, which by being based on marginal contributions pay player 3 a positive amount.

EXAMPLE 2. The NTU prekernel may be empty over the class considered. Let $N=\{1, 2, 3\}$; $V(\{i\})$ is the non-positive real half-line as before; $V(\{1, 2\})$ is the comprehensive hull of the convex hull of $(1, 0)$ and $(0, 2/3)$: $V(\{1, 2\})=\text{cch}[(1, 0), (0, 2/3)]$; similarly, $V(\{1, 3\})=\text{cch}[(2/3, 0), (0, 1)]$; and $V(\{2, 3\})=\text{cch}[(1, 0), (0, 2/3)]$; $V(N)$ is the set of payoffs that add up to no more than 1. Then, $\text{Prk}(N, V)=\varnothing$.

[16] This example is the center of the controversy between Roth [28, 29] and Aumann [1, 2].

Although a cyclical equilibrium exists (by the proof of Theorem 3 if we truncated $V(N)$), a singleton limit does not. To identify interesting classes of NTU games where the prekernel is non-empty is an open question (Moldovanu [23] proves existence for the class of assignment problems).

EXAMPLE 3. The NTU prekernel is not a selection of the Aumann–Maschler bargaining set. Let $N = \{1, 2, 3\}$. Let the sets $V(\{i\})$ for every i be the same as in the previous examples. For $j = 2, 3$, let $V(\{1, j\})$ be the comprehensive hull of $\{\mathbf{x} \in \mathbb{R}^{\{1, j\}}$ such that $x_1^2 + x_j^2 = 1\}$; $V(\{2, 3\}) = \{\mathbf{x} \in \mathbb{R}^{\{2, 3\}}$ such that $x_2 + x_3 \leqslant 1\}$; $V(N)$ is the set of vectors whose sum is no greater than $2^{1/2}$. Then, simple computations show that $\text{Prk}(N, V) = \{(0.671, 0.371, 0.371)\}$ approximately. However, this point does not belong to the Aumann–Maschler bargaining set: player 1 has an objection (say, against player 3) $(0.68, 0.73)$, to which player 3 cannot counterobject.

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