



Axiomatizations of neoclassical concepts for economies

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Abstract

We characterize the Pareto correspondence, the core and the Walras solution using the axioms of consistency, converse consistency and one-person rationality. Consistency and its converse are defined with respect to suitably constructed reduced economies for each case. Our results hold for the well-known class of coalitional production economies, which covers exchange economies as a particular case. The key reason to use this class is the observation that the reduction of an exchange economy yields a production economy. © 1998 Elsevier Science S.A. All rights reserved.

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1. Introduction

The consistency property in its several formulations has served to characterize most solution concepts in game theory. For example, Peleg and Tijs (1996) characterize the Nash equilibrium correspondence; Lensberg (1988) axiomatizes the Nash bargaining solution; Peleg (1985, 1986) characterizes the core of NTU and TU games respectively; Hart and Mas-Colell (1989) axiomatize the Shapley value and Sobolev (1975) characterizes the prenucleolus. In the context of private ownership economies, the Walrasian correspondence was characterized first by

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Dagan (1994) and later by van den Nouweland et al. (1996) for some specific class of generalized economies. In all these characterizations, the concept of consistency played a crucial role. In economies with a social endowment, Thomson (1988) and Thomson and Zhou (1993) show that the equal income Walrasian solution satisfies consistency.¹

Thomson (1995) gives the following informal description of consistency: “A solution is *consistent* if whenever it recommends x as the solution outcome of some admissible problem, then it recommends the restriction of x to any subgroup as solution outcome of the ‘reduced problem’ faced by this subgroup: this is the problem obtained from the original one by attributing to the members of the complementary subgroup ‘their components of x .’”

Of course, as is well-known, the definition of the ‘reduced problem’ is the key step. Roughly, one could distinguish two types of ‘reduced problems’ for a group of agents S . (1) One could imagine that those agents who are not in S left the scene with their components of x and further cooperation and/or interaction with them is no longer possible. (2) Alternatively, one could envision that potential cooperation with agents not in S is used as an argument for bargaining among the agents in S . That is, it is as if the agents not in S have signed a contract under which they are willing to supply their resources if they are paid x . Instances of (1) can be found in bankruptcy, where the standard ‘reduced problem’ specifies the amount of the estate left to the members of S after the complementary group abandoned the room with the amounts allocated to them. This approach is also followed by Dagan (1994) and van den Nouweland et al. (1996) in their definition of a reduced generalized economy. Examples of (2) are the reduced games introduced by Davis and Maschler (1965) and by Hart and Mas-Colell (1989). In these two reduced games, the worth of any subcoalition of S is defined taking into account the potential interactions with those agents who are not in S .

Following (2), this paper uses consistency and converse consistency to characterize some of the central solution concepts in economics. We introduce several variations of a reduced economy, which differ in the way the interaction between the agents in the reduced economy and those that stayed outside is modeled. Using these different reduced economies and consistency, we are able to characterize the Pareto correspondence, the core and a slight modification of the Walrasian correspondence, which we refer to as the Walras solution.²

Our characterizations apply to very general classes of economies. In particular, our characterizations of the core and the Pareto correspondence work for the

¹ For comprehensive surveys on consistency and its applications, the reader is referred to Thomson (1990, 1995).

² This solution departs from the Walrasian correspondence in that it selects all the individually rational allocations in one-person economies and all shrunk core allocations in economies with more than one agent. For the relation between Walrasian and shrunk core allocations, see Debreu and Scarf (1963).

universal class of all economies, while that of the Walras solution does for the class of additive coalitional economies.³ That is, our axiomatizations are not restricted to the class for which a given solution concept is non-empty. We regard these two issues (axiomatic foundation and the problem of existence) as unrelated.

Our reduced economies are inspired by the characterizations of Dagan (1996) of core and shrunk core allocations. Moreover, the proofs of our results rely heavily on this work. Dagan (1996) makes it clear that both core and shrunk core are expressions of agents' individual optimization (for a suitably defined feasible set) with coalitions playing no crucial role. Therefore, it is not surprising that one-agent reduced economies are central to our contribution (in this sense, our characterizations resemble the work of Peleg and Tijs (1996)).

The first reduced economy we study is based on the reduced game of Davis and Maschler (1965). The situation it describes is as follows. Suppose an allocation x has been proposed to the economy. In order to define the reduced economy for a group S of agents, we assume that the feasible set for S contains all the bundles that can be produced by the grand coalition and which remain after the agents not in S are given bundles that they (weakly) prefer to x , i.e., S must cooperate with all the agents who are not in S . On the other hand, subcoalitions of S are allowed to imagine that they could cooperate in this fashion with any subset of agents outside S . When we use this reduced economy, we characterize the Pareto correspondence with the axioms of one-person rationality, consistency and converse consistency; likewise, the core is axiomatized with one-person rationality, individual rationality, consistency and a weakening of converse consistency.⁴

One criticism usually raised against the D–M reduced game is that different subcoalitions may be using the resources of intersecting coalitions of agents outside S , thereby yielding feasibility problems. Countering this criticism, one can argue that the outside agents are used only to determine the relative bargaining power of the subcoalitions of S . Notice however that S itself can imagine this 'cooperation' with the outside agents without jeopardizing feasibility. Therefore, if one is willing to accept the logic of the D–M reduced economy for subcoalitions of S , there should be no problem accepting it for S itself. Our second reduced economy dispenses with the asymmetry of the Davis–Maschler reduced game. That is, in the reduced economy for S , the grand coalition is allowed to imagine the same kind of operations as its subcoalitions. When we use this reduced economy, the core obtains as the unique solution that satisfies one-person rationality, consistency and converse consistency.

³ Additive coalitional economies include pure exchange economies.

⁴ The axiomatizations of the core of games in coalitional form of Peleg (1985, 1986) use the Davis–Maschler reduced game. They are given for classes of games with non-empty cores and the crucial reduced games are bilateral.

Finally we pursue this ‘cooperation’ further. The third reduced economy we study allows its agents to cooperate *à la Davis–Maschler* infinitely-many times. That is, suppose that each agent’s production possibility set in the original economy is given. Suppose an allocation x has been proposed and we need to define the reduced economy for S . Then, each agent in S can, as before, ‘cooperate’ with the agents not in S . But this is just a first iteration, i.e., each of them imagines that he has a larger production possibility set as a result of these trades. Now these new sets can be taken as the basis for new trades, which would lead to a second iteration of sets, and so on ad infinitum. That is, we could imagine that these agents are connected through a computer network. When a group of agents wants to cooperate, each of their computers calculates the respective agent’s feasible set as a function of the resources of the other agents and sends back the result to the other computers. Using the new information, the computers can recalculate new feasible sets as a function of the updated resources of the other agents, and so on. The limit set is shown by the computer to the agent as this agent’s feasible set of bundles. It is as if these agents were inevitably thirsty of profits and nothing would stop them from trying to take advantage of an arbitrage opportunity. Of course, the problem is that these production sets exhibit infeasibilities almost everywhere: it may well be the case that the trade someone is offering me requires my own previous cooperation for a number of times. As Dagan (1996) shows, if the world were populated by this type of agents, the only allocations that could survive this insatiable thirst of profits are essentially the Walrasian ones. Moreover, the feasible sets that result from the iterative process described above are essentially budget sets generated by a common price vector. The axiomatic expression of this is our characterization of the Walras solution with one-person rationality, consistency and converse consistency by means of this reduced economy.⁵

In sum, this paper characterizes some of the central solution concepts in economics by means of consistency and converse consistency. Consistency can be interpreted as an axiom of ‘equilibrium’ (in the sense of a self-consistent allocation-expectations property). Namely, it ‘sustains’ an allocation by providing an exact description of the expectations that agents and coalitions have if they were to ‘deviate’ and reject the allocation in question. Converse consistency, on the other hand, states that the solution can be ‘decentralized’ by being imposed on smaller groups, where each of them holds the appropriate expectations, as described by the relevant reduced economy. The uncovering of the expectations that underlie different solution concepts contributes to our better understanding of them.

To obtain our results, it is crucial to properly define the reduced economies. The key step in the construct is to realize that the reduction of an exchange

⁵ Other characterizations of the Walrasian correspondence are found in Gevers (1986) and Nagahisa (1991, 1994). They are not consistency-based, however.

economy yields a production economy. Therefore, the natural class of economies to consider is the class of production economies which, unlike that of exchange economies, is closed under the reduction operation. Once this is understood, the rest follows essentially the approach taken by Peleg (1986, 1985) in his characterizations of the core. If one wishes to remain within the class of exchange economies, one is led to consider generalized economies as in Dagan (1994) and in van den Nouweland et al. (1996). These generalized economies, though, are invented to make a particular notion of consistency fit in. One additional advantage of our definition of reduced economies is that it yields ‘pure’ axiomatizations, in the sense that the class of economies to which they apply have nothing to do with the solution concepts characterized. This is in contrast to van den Nouweland et al. (1996) where the result holds for the class of generalized economies for which their solution concept is non-empty.

Another important feature of our characterization is that it does not rely on technical assumptions on the economy. In particular, we do not need to restrict attention to a class of economies with differentiable preferences. This is in contrast to Nagahisa (1991), Dagan (1994) and van den Nouweland et al. (1996). Our economies do not require even that agents’ preferences be representable by utility functions. Another feature of the result, which is related to the previous point, is that when considering a closed class of economies in which the Walrasian allocations coincide with the shrunk core allocations, our characterization of the Walras solution in fact characterizes the competitive equilibrium allocations without making use of prices. In other words, prices are a consequence of the axioms and not an auctioneer’s tool.

The paper is organized as follows: Section 2 presents basic notation and definitions. Section 3 analyzes the Davis–Maschler reduced economy and presents characterizations of the Pareto correspondence and the core. A slight modification of the Davis–Maschler reduced economy is presented in Section 4, where a simple characterization of the core is given (we choose to begin with the Davis–Maschler reduced economy for historical reasons. Readers interested in characterizations of the core should perhaps skip Section 3 and read Section 4 first). Section 5 studies additive coalitional economies and characterizes the Walras solution.

2. Preliminaries

2.1. Notation

We denote by \mathbb{R}^l the l -dimensional Euclidean space. Given two non-empty subsets A and B of \mathbb{R}^l , we shall denote by $A + B = \{x \in \mathbb{R}^l \mid \exists a \in A, b \in B \text{ such that } x = a + b\}$. We shall denote by $A - B$ the set $A + (-B)$. We also postulate that $A + \emptyset = A$. We denote by \mathbb{N} the set of natural numbers.

2.2. Definitions

An economy is a system $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$, where N is a finite set of agents; for each agent $i \in N$, $X_i \subseteq \mathbb{R}^l$ is i 's consumption set, and \succeq_i is his reflexive preference relation over bundles in X_i ; and for each $S \subseteq N$, $S \neq \emptyset$, $Y_S \subseteq \mathbb{R}^l$ is the production possibilities set of coalition S .⁶

These economies were introduced in Boehm (1974). They include private ownership economies as a particular case.

We refer to non-empty subsets S of N as coalitions, and to non-empty proper subsets S of N as proper coalitions. Let S be a coalition. An S -allocation in \mathcal{E} is a list of bundles $(x_i)_{i \in S}$ such that $x_i \in X_i \ \forall i \in S$ and $\sum_{i \in S} x_i \in Y_S$. We shall refer to N -allocations simply as allocations and we denote the set of allocations in \mathcal{E} by $\mathcal{A}(\mathcal{E})$.

For every $i \in N$ and $x_i \in X_i$, define the preferred and the weakly preferred sets as follows:

$$P_i(x_i) = \{z_i \in X_i \mid z_i \succ_i x_i\}$$

$$W_i(x_i) = \{z_i \in X_i \mid z_i \succeq_i x_i\}.$$

where \succ_i is i 's strict preference relation and it is defined as usual.

An allocation $(x_i)_{i \in N}$ in \mathcal{E} is said to be *improved upon* by a coalition S if there exists $i \in S$ with

$$P_i(x_i) \cap \left[Y_S - \sum_{k \in S \setminus \{i\}} W_k(x_k) \right] \neq \emptyset.$$

An allocation in \mathcal{E} is said to be *individually rational* if it cannot be improved upon by any singleton coalition.

An allocation in \mathcal{E} is *efficient* if it cannot be improved upon by the grand coalition N .

An allocation in \mathcal{E} is said to be *coalitionally rational* if it cannot be improved upon by any coalition.

Let E be a class of economies. A *solution on E* is a set-valued function ϕ that assigns to each economy $\mathcal{E} \in E$ a set of allocations in \mathcal{E} .

Examples: Let E_0 be the class of all economies.

1. The core \mathcal{C} , is the solution that assigns to each economy the set of its coalitionally rational allocations.
2. The Pareto optimal solution $\mathcal{P}\mathcal{O}$, assigns to each economy the set of its efficient allocations.

⁶ In fact, since all our arguments are purely set theoretic, we believe that all the results would go through if consumption and production sets were subsets of infinite dimensional spaces. Unlike the consideration of production economies, infinite dimensional spaces are not needed for the results and we choose our simpler formulation.

3. The individually rational solution \mathcal{IR} , assigns to each economy the set of its individually rational allocations.
4. The empty solution assigns to each economy the empty set.
5. The Walrasian equilibrium solution \mathcal{WE} assigns to each economy the set of all allocations x for which there exists a price vector $p \in \mathbb{R}_+^l \setminus \{0\}$ such that for all $i \in N$

$$P_i(x_i) \cap \{z \in \mathbb{R}_+^l : pz \leq py \ \forall y \in Y_i\} = \emptyset$$

Two rather weak properties of solutions, which are expressions of the rationality of the agents, are:

A solution ϕ on a class E of economies satisfies *one-person rationality* (OPR) if it assigns to each one-person economy in E the set of its individually rational allocations.

A solution ϕ on a class E of economies satisfies *individual rationality* (IR) if it assigns to each economy in E a subset of its individually rational allocations, i.e., $\phi(\mathcal{E}) \subseteq \mathcal{IR}(\mathcal{E})$ for all $\mathcal{E} \in E$

3. The Davis–Maschler reduced economy

In this section we shall introduce our first type of reduced economy and define its related properties of consistency and converse consistency. We further use these properties to characterize the Pareto correspondence and the core.

Definition 1: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy. Let $x = (x_i)_{i \in N}$ be an allocation in \mathcal{E} and let S be a coalition. The *reduced economy à-la-Davis–Maschler* with respect to x and S is defined as follows:

$$\mathcal{E}^{S,x} = \langle S, (X_i, \succeq_i)_{i \in S}, (Y_T^{S,x})_{\emptyset \neq T \subseteq S} \rangle$$

where

$$Y_T^{S,x} = \begin{cases} Y_N - \sum_{i \in N \setminus S} W_i(x_i) & \text{if } T = S \\ \bigcup_{F \subseteq N \setminus S} [Y_{T \cup F} - \sum_{i \in F} W_i(x_i)] & \text{if } T \subset S, T \neq S \end{cases}$$

In this reduced economy, the agents in S are committed to interact with all the agents in $N \setminus S$ under the premise that each of them will receive a bundle which is not worse than their components of x . Therefore, the members of S are allowed only to redistribute among themselves bundles that are compatible with such a commitment. On the other hand, proper coalitions of S may choose any subset of $N \setminus S$ in order to provide an argument to strengthen their bargaining position in the reduced economy. In other words, each of these coalitions imagines that they can leave the reduced economy and cooperate with some of the subsets of $N \setminus S$. The reader will recognize in this definition the spirit of the Davis–Maschler reduced game, applied to this setup.

A class E of economies is said to be *closed under the reduction operation* (or closed, for short), if for every economy $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ in E, for every coalition $S \subseteq N$, $S \neq \emptyset$ and for every allocation $(x_i)_{i \in N} \in \mathcal{A}(\mathcal{E})$ we have $\mathcal{E}^{S,x} \in E$.

The class of all the economies is clearly closed under the reduction operation.

A solution ϕ on a class E of economies satisfies *DM-consistency* (DM-CONS) if for every economy $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ in E, for every $S \subseteq N$, $S \neq \emptyset$, if $(x_i)_{i \in N} \in \phi(\mathcal{E})$, then $(x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$.

Consistency requires that if x is prescribed by ϕ for an economy \mathcal{E} , then the projection of x to S should be prescribed by ϕ for the reduced economy with respect to S and x for all coalitions S . Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced economy.

A solution ϕ on a class E of economies satisfies *DM-converse consistency* (DM-COCONS) if the following condition holds: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy in E, and let $(x_i)_{i \in N}$ be an efficient allocation in \mathcal{E} . If $(x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$, for all $S \subseteq N$, $S \neq \emptyset$, N , then $(x_i)_{i \in N} \in \phi(\mathcal{E})$.

Converse consistency requires that if the projection of an efficient allocation x to every proper coalition S is consistent with the expectations of the members of S as reflected by their reduced economy then x itself should be recommended for the whole economy.

A solution ϕ on a class E of economies satisfies *weak converse consistency* (WCOCONS) if the following condition holds: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy in E, and let $(x_i)_{i \in N}$ be an individually rational and efficient allocation in \mathcal{E} . If $(x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$, for all $S \subseteq N$, $S \neq \emptyset$, N , then $(x_i)_{i \in N} \in \phi(\mathcal{E})$.

This is a weakening of the previous definition, since it requires that x be individually rational as well.

We state now our first result:

Theorem 1: *A solution ϕ on a closed class E satisfies OPR, DM-CONS and DM-COCONS if and only if $\phi = \mathcal{PO}$ (i.e., $\phi(\mathcal{E}) = \mathcal{PO}(\mathcal{E})$ for every economy $\mathcal{E} \in E$).*

Proof: Let E be a closed class of economies. The reader can check that \mathcal{PO} satisfies the three foregoing axioms. To prove the Theorem, we need to show the opposite implication. This will follow from the two lemmas below.

Lemma 1 *Let ϕ be a solution on a class E that satisfies OPR and DM-CONS. For all $\mathcal{E} \in E$, if $x \in \phi(\mathcal{E})$, then x is an efficient allocation in \mathcal{E} .*

Proof: Let $(x_i)_{i \in N} \in \phi(\mathcal{E})$. By DM-consistency of ϕ , $x_i \in \phi(\mathcal{E}^{(i),x}) \forall i \in N$. By OPR, $P_i(x_i) \cap Y_i^{(i),x} = \emptyset \forall i \in N$. But since $Y_i^{(i),x} = Y_N - \sum_{k \in N \setminus \{i\}} W_k(x_k)$, it follows that $(x_i)_{i \in N}$ is efficient in \mathcal{E} .

Remark: In fact, the careful reader will note that the above lemma holds with independence of the way the reduced economy is defined, as long as $Y_i^{(i),x} \supseteq Y_N - \sum_{k \in N \setminus \{i\}} W_k(x_k)$. This fact will be used in the sections below.

Lemma 2: Let φ be a DM-consistent and OPR solution on E and let ψ be DM-converse consistent on the same class. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies \mathcal{E} , then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all economies \mathcal{E} in E .

Proof ⁷: We prove the Lemma by induction. By assumption, the claim is true for all one-person economies. Suppose next it holds for all k -person economies, with $1 \leq k \leq n - 1$ and let \mathcal{E} be an n -person economy. Let $(x_i)_{i \in N} \in \varphi(\mathcal{E})$. By Lemma 1, $(x_i)_{i \in N}$ is an efficient allocation. By DM-consistency of φ , $(x_i)_{i \in S} \in \varphi(\mathcal{E}^{S,x})$ for all $S \subseteq N$, $S \neq \emptyset$, N . By the induction hypothesis, $(x_i)_{i \in S} \in \psi(\mathcal{E}^{S,x})$ for all $S \subseteq N$, $S \neq \emptyset$, N . By DM-converse consistency of ψ , $(x_i)_{i \in N} \in \psi(\mathcal{E})$.

Remark: Note again that this lemma holds with independence of the way in which the reduced economy is defined.

Let now ϕ be a solution on E that satisfies OPR, DM-CONS and DM-COCONS. Since both ϕ and $\mathcal{P}\mathcal{O}$ satisfy OPR, they coincide for all one-person economies. Further, since both satisfy DM-CONS and DM-COCONS, by Lemma 2, they must coincide for all economies.

The following examples show that the axioms used in the characterization are independent:

Example 1: The empty solution satisfies DM-CONS and DM-COCONS, but violates OPR.

Example 2: Let ϕ be a solution on E_0 defined as follows:

$$\phi(\mathcal{E}) = \begin{cases} \mathcal{I}\mathcal{R}(\mathcal{E}) & \text{if } |N| = 1 \\ \mathcal{A}(\mathcal{E}) & \text{otherwise} \end{cases}$$

It can be easily checked that ϕ satisfies OPR and DM-COCONS. Since $\phi \neq \mathcal{P}\mathcal{O}$, by Theorem 1, it does not satisfy DM-CONS (for a direct proof, choose a non-efficient allocation in a two-person economy).

Example 3: It will be shown in Theorem 3 that the core on E satisfies OPR and DM-CONS. Since $\mathcal{C} \neq \mathcal{P}\mathcal{O}$, by Theorem 1, it does not satisfy DM-COCONS (for a direct proof, choose an efficient allocation in a two-person economy which is not individually rational).

⁷ This extremely powerful and simple lemma was shown to us by Stef Tijs.

The following definition shows an example of an abstract equilibrium. Abstract equilibria specify for each agent in the economy a bundle and a feasible set, and requires (a) that the bundles constitute an allocation in the economy, and (b) that each agent’s bundle be maximal in the agent’s preference ordering within the feasible set. See Dagan (1996) and the references therein for more details.

Definition 2: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy. An E-equilibrium is a collection $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ such that:

- (i) $(x_i)_{i \in N}$ is an allocation in \mathcal{E} ;
- (ii) $Y_S - \sum_{k \in S \setminus \{i\}} W_k(x_k) \subseteq C_i \quad \forall i \in S, \forall S \subseteq N$;
- (iii) $P_i(x_i) \cap C_i = \emptyset \quad \forall i \in N$.

The special feature of the E-equilibrium as an abstract one is the ‘recontracting’ conditions (ii). The following is a simple extension of Theorem 3 in Dagan (1996). We omit its proof. (Dagan’s result does not apply directly to our class of economies, but to exchange economies and to a class of production economies different from the ones in this paper):

Theorem 2: Let \mathcal{E} be an economy. An allocation $(x_i)_{i \in N}$ belongs to the core of \mathcal{E} if and only if it can be supported by an E-equilibrium. Moreover, the sets $(C_i)_{i \in N}$ can be chosen to be the following:

$$C_i = \bigcup_{\substack{S \subseteq N \\ i \in S}} \left[Y_S - \sum_{k \in S \setminus \{i\}} W_k(x_k) \right], i \in N. \tag{1}$$

Next we use this theorem to obtain the following characterization.

Theorem 3: A solution ϕ on a closed class E satisfies OPR, IR, DM-CONS and WCOCONS if and only if $\phi = \text{Core}$ (i.e., $\phi(\mathcal{E}) = \mathcal{C}(\mathcal{E})$ for every economy $\mathcal{E} \in E$).

Proof: Let E be a closed class of economies. First we show that the core on E satisfies OPR, IR, DM-CONS and WCOCONS. It is well-known that the core satisfies OPR and IR.

To see that the core is DM-consistent, let \mathcal{E} be an economy in E and let $(x_i)_{i \in N} \in \mathcal{C}(\mathcal{E})$. By Theorem 2, $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is an E-equilibrium where

$$C_i = \bigcup_{\substack{S \subseteq N \\ i \in S}} \left[Y_S - \sum_{k \in S \setminus \{i\}} W_k(x_k) \right]. \tag{2}$$

Let $S \subseteq N$ be a coalition. We need to show that $(x_i)_{i \in S} \in \mathcal{C}(\mathcal{E}^{S,x})$. By Theorem 2, it is enough to show that $\langle (x_i)_{i \in S}, (C_i^{S,x})_{i \in S} \rangle$ is an E-equilibrium where

$$C_i^{S,x} = \bigcup_{\substack{F \subseteq S \\ i \in F}} \left[Y_F^{S,x} - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right], i \in S. \tag{3}$$

First note that $\sum_{i \in S} x_i \in [Y_N - \sum_{k \in N \setminus S} W_k(x_k)]$ because $(x_i)_{i \in N}$ is an allocation in \mathcal{E} and $x_k \in W_k(x_k)$ for all $k \in N \setminus S$. Therefore, $(x_i)_{i \in S} \in Y_S^{S,x}$. Second, by definition of $C_i^{S,x}$, (ii) in the definition of an E-equilibrium is satisfied.

Condition (iii) follows from the lemma below:

Lemma 3: Let $\mathcal{E} = \langle N, (X_i, \geq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N}$ be an economy and let $i \in N$. Then: (a)

If $|N| \geq 3$, then

$$\bigcup_{\substack{S \subseteq N \\ i \in S}} C_i^{S,x} = C_i;$$

(b) If $|N| = 2$, then $C_i^{(i),x} \subseteq C_i$ and $C_i \setminus C_i^{(i),x} \subset Y_i$.

Proof: Fix an agent i and let $i \in S \subseteq N$.

(a) By definition of $C_i^{S,x}$,

$$C_i^{S,x} = \bigcup_{\substack{F \subseteq S \\ i \in F}} \left[Y_F^{S,x} - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right].$$

Therefore,

$$\bigcup_{\substack{S \subseteq N \\ i \in S}} C_i^{S,x} = \bigcup_{\substack{S \subseteq N \\ i \in S}} \left\{ \bigcup_{\substack{F \subseteq S \\ i \in F}} \left[Y_F^{S,x} - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right] \right\},$$

which by definition of $Y_F^{S,x}$ (for $F = S$ and for $F \subset S$) equals

$$\bigcup_{\substack{S \subseteq N \\ i \in S}} \left\{ \left[Y_N - \sum_{k \in N \setminus \{i\}} W_k(x_k) \right] \cup \left\{ \bigcup_{\substack{F \subseteq S \\ i \in F}} \left[\bigcup_{Q \subseteq N \setminus S} \left(Y_{F \cup Q} - \sum_{k \in Q \setminus S} W_k(x_k) \right) - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right] \right\} \right\}.$$

But this can be written as:

$$\bigcup_{\substack{R \subseteq N \\ i \in R}} \left[Y_R - \sum_{k \in R \setminus \{i\}} W_k(x_k) \right],$$

which is precisely C_i .

(b) Let $N = \{i, j\}$. By definition, for each $i \in N$ we have

$$\begin{aligned} C_i^{(j),x} &= Y_N - W_j(x_j) && i \neq j \\ &\subseteq [Y_N - W_j(x_j)] \cup Y_i && i \neq j. \\ &= C_i \end{aligned}$$

□

Since $P_i(x_i) \cap C_i = \emptyset$ for all $i \in S$ and since by Lemma 3 $C_i^{S,x} \subseteq C_i$ for all $i \in S$, we have that $P_i(x_i) \cap C_i^{S,x} = \emptyset$, for all $i \in S$. Since S was chosen arbitrarily, this implies that (iii) is satisfied.

Next we show that the core satisfies WCOCONS. Let \mathcal{E} be an economy with at least two agents (if it is a one-person economy, there is nothing to be proved), and let $(x_i)_{i \in N}$ be an individually rational and efficient allocation in \mathcal{E} . Assume that for all $S \subseteq N$, $S \neq \emptyset$, N , $(x_i)_{i \in S} \in \mathcal{E}(\mathcal{E}^{S,x})$. We must show that $(x_i)_{i \in N} \in \mathcal{E}(\mathcal{E})$. By Theorem 2, it is enough to show that $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is an E-equilibrium for C_i defined in Eq. (2). Condition (i) is satisfied by assumption. By definition of C_i , condition (ii) is immediately satisfied as well.

To show (iii), let $S \subseteq N$, $S \neq \emptyset$, be a proper coalition. By assumption and by Theorem 2, $\langle (x_i)_{i \in S}, (C_i^{S,x})_{i \in S} \rangle$ is an E-equilibrium for $\mathcal{E}^{S,x}$. Hence, it must satisfy (iii), that is,

$$P_i(x_i) \cap C_i^{S,x} = \emptyset, \forall i \in S. \tag{4}$$

If $|N| \geq 3$, by Lemma 3, this amounts to $P_i(x_i) \cap C_i = \emptyset$ for all $i \in S$. If $|N| = 2$, note that

$$P_i(x_i) \cap C_i = P_i(x_i) \cap [(C_i \cap C_i^{(j),x}) \cup (C_i \setminus C_i^{(j),x})],$$

which by Lemma 3 part (b), is included in

$$[P_i(x_i) \cap C_i^{(j),x}] \cup [P_i(x_i) \cap Y_i].$$

By Eq. (4) and by individual rationality of the allocation $(x_i)_{i \in N}$, the two components of this union are empty. Since S was chosen arbitrarily, this shows that (iii) is satisfied. Therefore, as claimed, the core satisfies WCOCONS.

To prove the Theorem, we need to show the opposite implication. This will follow from the next lemma.

Lemma 4: *Let φ be a solution on E that satisfies OPR, IR and DM-CONS, and let ψ be a solution on the same class that satisfies WCOCONS. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies \mathcal{E} , then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all economies \mathcal{E} in E .*

Proof: We prove the Lemma by induction. By assumption, the claim is true for all one-person economies. Suppose next it holds for all k -person economies, with

$1 \leq k \leq n - 1$ and let \mathcal{E} be an n -person economy. Let $(x_i)_{i \in N} \in \varphi(\mathcal{E})$. By individual rationality of φ , $(x_i)_{i \in N}$ is an individually rational allocation. By Lemma 1, $(x_i)_{i \in N}$ is an efficient allocation. By DM-consistency of φ , $(x_i)_{i \in S} \in \varphi(\mathcal{E}^{S,x})$ for all $S \subseteq N$, $S \neq \emptyset$, N . By the induction hypothesis, $(x_i)_{i \in S} \in \psi(\mathcal{E}^{S,x})$ for all $S \subseteq N$, $S \neq \emptyset$, N . By weak converse consistency of ψ , $(x_i)_{i \in N} \in \psi(\mathcal{E})$.

Let now ϕ be a solution on E that satisfies OPR, IR, CONS and WCOCONS. Since both ϕ and \mathcal{E} satisfy OPR, they coincide for all one-person economies. Further, since both satisfy IR, DM-CONS and WCOCONS, by Lemma 4, they must coincide for all economies. \square

The following examples show that the axioms used in the characterization are independent:

Example 4: The empty solution satisfies IR, DM-CONS and WCOCONS, but violates OPR.

Example 5: The $\mathcal{P}\mathcal{O}$ solution satisfies OPR, DM-CONS and WCOCONS, but violates IR.

Example 6: The $\mathcal{I}\mathcal{R}$ solution satisfies OPR, IR and WCOCONS. Since $\mathcal{I}\mathcal{R} \neq \mathcal{E}$, by Theorem 3, it violates DM-CONS (for a direct proof, choose an individually rational but non-efficient allocation in a two-person economy).

Example 7: Consider the class E_M of economies with strictly monotone preferences (which is closed). Let ϕ be a solution on E_M defined as follows:

$$\phi(\mathcal{E}) = \begin{cases} \mathcal{I}\mathcal{R}(\mathcal{E}) & \text{if } |N| = 1; \\ \mathcal{W}\mathcal{E}(\mathcal{E}) & \text{otherwise} \end{cases}$$

It can be checked that ϕ satisfies OPR, IR and DM-CONS, but does not satisfy WCOCONS.

Remark: Theorem 3 is related to the characterization of the NTU core by Peleg (1985). There are some important differences, however. First, Peleg’s result holds for either a variable number of agents or for a finite set of agents of cardinality larger than 2. Our result, on the other hand, does not impose such restrictions. Second, while Peleg’s result applies to a specific closed class of games, that where the core is non-empty, ours holds for any closed class of economies. In particular, it holds for the class of all economies. Consequently, our characterization yields the empty set when the core is empty. In addition, Peleg (1985) also requires non-emptiness of the solution as an axiom. In contrast, we do not require this: we believe that the axioms behind a solution should stay with it even when the solution is empty. Third, unlike Peleg’s, our result does not require any technical assumption on the economy. In particular, individuals’ preferences need not have a

numerical representation. Having said all this, we should make it clear that the roots of our result are found in Peleg’s.

4. An alternative reduced economy

This section modifies the reduced economy used earlier and presents consistency and converse consistency accordingly. These new axioms are then utilized to provide an alternative characterization of the core.

Definition 3: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy. Let $x = (x_i)_{i \in N}$ be an allocation in \mathcal{E} and let S be a coalition. The *DM’-reduced economy* with respect to x and S is defined as follows:

$$\mathcal{E}^{S,x} \langle S, (X_i, \succeq_i)_{i \in S}, (Y_T^{S,x})_{\emptyset \neq T \subseteq S} \rangle$$

where

$$Y_T^{S,x} = \bigcup_{F \subseteq N \setminus S} \left[Y_{T \cup F} - \sum_{i \in F} W_i(x_i) \right], T \subseteq S.$$

The only difference between this and the D–M reduced economy is the fact that the coalition S is also allowed to imagine potential interaction with any of the subsets of $N \setminus S$. In other words, they are not required to cooperate with all the members of $N \setminus S$. Note that the usual criticism to the D–M reduced economy-i.e., the fact that two coalitions may imagine mutually incompatible interactions-does not apply to S . The definition of a closed class of economies given in Section 2 applies to this kind of reduced economies as well.

A solution ϕ on a class E of economies satisfies *DM’-consistency* (DM’-CONS) if for every economy $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ in E , for every $S \subseteq N$, $S \neq \emptyset$, if $(x_i)_{i \in N} \in \phi(\mathcal{E})$, then $(x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$.

A solution ϕ on a class E of economies satisfies *DM’-converse consistency* (DM’-COCONS) if the following condition holds: Let $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ be an economy in E , and let $(x_i)_{i \in N}$ be an efficient allocation in \mathcal{E} . If $(x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$, for all $S \subseteq N$, $S \neq \emptyset$, N , then $(x_i)_{i \in N} \in \phi(\mathcal{E})$.

The interpretations of the above two properties are similar to those given in Section 3. Our next characterization result follows.

Theorem 4: A solution ϕ on a closed class E satisfies OPR, DM’-CONS and DM’-COCONS if and only if $\phi = \text{Core}$ (i.e., $\phi(\mathcal{E}) = \mathcal{C}(\mathcal{E})$ for every economy $\mathcal{E} \in E$).

Proof: Let E be a closed class of economies. First we show that the core on E satisfies OPR, DM’-CONS and DM’-COCONS. It is easy to see that the core satisfies OPR.

To see that the core is DM'-consistent, let \mathcal{E} be an economy in E and let $(x_i)_{i \in N} \in \mathcal{C}(\mathcal{E})$. By Theorem 2, $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is an E-equilibrium where

$$C_i = \bigcup_{\substack{S \subseteq N \\ i \in S}} \left[Y_S - \sum_{k \in S \setminus \{i\}} W_k(x_k) \right]. \tag{5}$$

Let $S \subseteq N$ be a coalition. We need to show that $(x_i)_{i \in S} \in \mathcal{C}(\mathcal{E}^{S,x})$. By Theorem 2, it is enough to show that $\langle (x_i)_{i \in S}, (C_i^{S,x})_{i \in S} \rangle$ is an E-equilibrium where

$$C_i^{S,x} = \bigcup_{\substack{S \subseteq N \\ i \in S}} \left[Y_S^{S,x} - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right] \quad i \in S. \tag{6}$$

First note that $\sum_{i \in S} x_i \in [Y_N - \sum_{k \in N \setminus S} W_k(x_k)]$ because $(x_i)_{i \in N}$ is an allocation in E and $x_k \in W_k(x_k)$. Therefore, $(x_i)_{i \in S} \in Y_S^{S,x}$. Second, by definition of $C_i^{S,x}$, (ii) in the definition of an E-equilibrium is satisfied.

Condition (iii) follows from the following lemma:

Lemma 5: $C_i^{S,x} = C_i$

Proof: By definition of $C_i^{S,x}$,

$$C_i^{S,x} = \bigcup_{\substack{F \subseteq S \\ i \in F}} \left[Y_F^{S,x} - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right].$$

By definition of $Y_F^{S,x}$, the production possibilities set of F in the reduced economy with respect to S and x , the latter expression equals

$$\bigcup_{\substack{F \subseteq S \\ i \in F}} \left[\bigcup_{Q \subseteq N \setminus S} \left(Y_{F \cup Q} - \sum_{k \in Q} W_k(x_k) \right) - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right].$$

But this can be written as:

$$\bigcup_{\substack{R \subseteq N \\ i \in R}} \left[Y_R - \sum_{k \in R \setminus \{i\}} W_k(x_k) \right],$$

which is precisely C_i . □

Next we show that the core satisfies DM'-COCONS. Let \mathcal{E} be an economy with at least two agents (if it is a one-person economy, there is nothing to be proved), and let $(x_i)_{i \in N}$ be an efficient allocation in \mathcal{E} . Assume that for all $S \subseteq N$, $S \neq \emptyset$, N , $(x_i)_{i \in S} \in \mathcal{C}(\mathcal{E}^{S,x})$. We must show that $(x_i)_{i \in N} \in \mathcal{C}(\mathcal{E})$. By Theorem 2, it is enough to show that $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is an E-equilibrium for C_i defined in Eq. (5).

Condition (i) is satisfied by assumption. By definition of C_i , condition (ii) is immediately satisfied as well. Let $S \subseteq N$, $S \neq \emptyset$, be a proper coalition. By

assumption and by Theorem 2, $\langle (x_i)_{i \in S}, (C_i^{S,x})_{i \in S} \rangle$ is an E-equilibrium for $\mathcal{E}^{S,x}$. Hence, it must satisfy (iii), that is, $P_i(x_i) \cap C_i^{S,x} = \emptyset$ for all $i \in S$. But by Lemma 5, this amounts to, $P_i(x_i) \cap C_i = \emptyset$ for all $i \in S$. Since S was chosen arbitrarily, this shows that (iii) is satisfied. Therefore, as claimed, the core satisfies DM'-COCONS.

To prove the Theorem, we need to show the opposite implication. This will follow from the next two lemmas.

Lemma 6: *Let ϕ be a solution on a class E that satisfies OPR and DM'-CONS. Then, $\phi \subseteq \mathcal{P}\mathcal{O}$, i.e., for all $\mathcal{E} \in E$ if $x \in \phi(\mathcal{E})$, then x is an efficient allocation in \mathcal{E} .*

Proof: It is similar to that of Lemma 1 taking into account its subsequent remark and is left to the reader. \square

Lemma 7: *Let φ be a DM'-consistent and OPR solution on E and let ψ be DM'-converse consistent on the same class. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies \mathcal{E} , then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all economies \mathcal{E} in E .*

Proof: It is analogous to that of Lemma 2 after taking into account its subsequent remark. \square

Let now ϕ be a solution on E that satisfies OPR, DM'-CONS and DM'-COCONS. Since both ϕ and \mathcal{E} satisfy OPR, they coincide for all one-person economies. Further, since both satisfy DM'-CONS and DM'-COCONS, by Lemma 7, they must coincide for all economies. \square

To demonstrate that the axioms used in the characterization are independent, consider the following examples.

Example 8: The empty solution satisfies DM'-CONS and DM'-COCONS, but does not satisfy OPR.

Example 9: Consider again the closed class E_M of economies with strictly monotone preferences. Let ϕ be defined on this class as follows:

$$\phi(\mathcal{E}) = \begin{cases} \mathcal{I}\mathcal{R}(\mathcal{E}) & \text{if } \mathcal{E} \text{ is a one - person economy.} \\ \mathcal{W}\mathcal{E}(\mathcal{E}) & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ satisfies OPR and DM'-CONS. By Theorem 4, it cannot satisfy DM'-COCONS because $\phi \neq \mathcal{E}$.

Example 10: It is easy to see that $\mathcal{P}\mathcal{O}$ satisfies OPR and DM'-COCONS. By Theorem 4, it cannot satisfy DM'-CONS because $\mathcal{P}\mathcal{O} \neq \mathcal{E}$. (For a direct proof, take an efficient allocation in a three-person economy which is improved upon by a two-person coalition S and consider their corresponding reduced economy).

5. Additive coalitional economies and the Walras solution

We now introduce a new modification of the reduced economy and explore its associated properties of consistency and converse consistency. They are then used to characterize the Walras solution on an interesting subclass of economies.

An economy $\mathcal{E} = \langle N, (X_i, \succeq_i)_{i \in N}, (Y_S)_{\emptyset \neq S \subseteq N} \rangle$ is said to be *additive* if $Y_S = \sum_{i \in S} Y_i \quad \forall S \subseteq N$. Since for an additive economy, in order to know the production possibilities of a coalition it is enough to know the production possibilities of each of its individual members, in this section we abuse notation slightly and denote a typical economy by $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$.

Denote by E_1 the class of all additive economies. For an additive economy we can define its replica economies as follows. For every $m \in \mathbb{N}$, let $\bar{m} = \{1, 2, \dots, m\}$. Let $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$ be an economy in E_1 and let $x = (x_i)_{i \in N}$ be an allocation in \mathcal{E} . The m -replica of \mathcal{E} is the economy $\mathcal{E}^m = \langle N \times \bar{m}, (X_{(i,j)}, \succeq_{(i,j)}, Y_{(i,j)})_{(i,j) \in N \times \bar{m}} \rangle$, where for all $(i,j) \in N \times \bar{m}$, $X_{(i,j)} = X_i$, $\succeq_{(i,j)} = \succeq_i$, and $Y_{(i,j)} = Y_i$. The m -replica of x is $x^m = (x_{(i,j)})_{(i,j) \in N \times \bar{m}}$, where $x_{(i,j)} = x_i \quad \forall (i,j) \in N \times \bar{m}$.

An allocation x is a *shrunk core allocation* if x^m is a core allocation of $\mathcal{E}^m \quad \forall m \in \mathbb{N}$.

5.1. Examples of solutions on E_1

The solutions $\mathcal{E}, \mathcal{F}\mathcal{R}, \mathcal{P}\mathcal{O}$ and the empty solution on E_1 are defined in a similar way as they were defined in Section 2. (1) The shrunk core $\mathcal{S}\mathcal{C}$ assigns to each additive economy the set of all its shrunk core allocations. (2) The Walras solution \mathcal{W} assigns to each one-person economy \mathcal{E} the set $\mathcal{W}(\mathcal{E}) = \mathcal{F}\mathcal{R}(\mathcal{E})$ and to each economy \mathcal{E} with at least two agents the set $\mathcal{W}(\mathcal{E})$ of all its shrunk core allocations.

Let $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$ be an economy. Let $x = (x_i)_{i \in N}$ be an allocation in \mathcal{E} and let $Q \subseteq N$ be a coalition. For any $i \in Q$ define inductively:

$$\begin{aligned} Z_i^{Q,x}(0) &= Y_i \\ Z_i^{Q,x}(t) &= \bigcup_{F \subseteq Q, i \in F} \left[\sum_{k \in F} Z_k^{Q,x}(t-1) - \sum_{k \in F \setminus \{i\}} W_k(x_k) \right] \\ Z_i^{Q,x} &= \bigcup_{t \in \mathbb{N}} Z_i^{Q,x}(t). \end{aligned}$$

That is, $Z_i^{Q,x}(0)$ represents agent i 's initial resources. As in the previous reduced economies, $Z_i^{Q,x}(1)$ represents the set of bundles that agent i could achieve by ‘cooperating’ with the members of any of the subsets of Q , where ‘cooperating’ with an agent j means purchasing j 's resources in exchange for a bundle that is no worse than x_j in the eyes of j . The set $Z_i^{Q,x}(t)$ has an analogous interpretation, provided that j 's resources are taken to be $Z_j^{Q,x}(t-1)$. The limit of this imaginary process leads to the sets $Z_i^{Q,x}$.

For example, consider a two-person, two-good exchange economy, where agents have identical preferences represented by the utility function $u(x) = \min\{x_1, x_2\}$ and endowments given by the bundles (2,4) and (4,2), respectively. Consider the allocation x that assigns the bundle (3,3) to each agent. It is easy to see that $Z_1^{N,x}(0) = \{(2,4)\}$ and $Z_2^{N,x}(0) = \{(4,2)\}$. Likewise, it can be easily checked that $Z_1^{N,x}(1) = \{(2, 4)\} \cup \{z \in \mathbb{R}_+^2 : z \leq (3,3)\}$ and $Z_2^{N,x}(1) = \{(4, 2)\} \cup \{z \in \mathbb{R}_+^2 : z \leq (3, 3)\}$ Further, it can be checked that

$$Z_1^{N,x}(2) = \{(2,4)\} \cup \{z \in \mathbb{R}_+^2 : z \leq (3,3) \text{ or } z \leq (2,4) \text{ or } z \leq (4,2)\}$$

and

$$Z_2^{N,x}(2) = \{(4,2)\} \cup \{z \in \mathbb{R}_+^2 : z \leq (3,3) \text{ or } z \leq (2,4) \text{ or } z \leq (4,2)\}$$

The limit of this process gives the sets $Z_i^{N,x}$.

Definition 4: Let $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$ be an economy. Let $x = (x_i)_{i \in N}$ be an allocation in \mathcal{E} and let S be a coalition. The DM^∞ -reduced economy with respect to x and S is defined as follows:

$$\mathcal{E}^{S,x} = \langle S, (X_i, \succeq_i, Y_i^{S,x})_{i \in S} \rangle$$

where

$$Y_i^{S,x} = \bigcup_{F \subseteq N \setminus S} [Z_i^{(i) \cup F, x}] \quad \forall i \in S.$$

The main difference between this reduced economy and that of Section 4 is the concept of ‘cooperation.’ When calculating the initial resources of an agent i in the reduced economy with respect to S and x , agent i is allowed to ‘cooperate’ with members of $N \setminus S$, where ‘cooperating’ has the interpretation we gave above.

A solution ϕ on E_1 is DM^∞ -consistent (DM^∞ -CONS) if for every economy $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$, if $x = (x_i)_{i \in N} \in \phi(\mathcal{E})$, then for every $S \subseteq N, S \neq \emptyset, (x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$.

A solution ϕ on E_1 is DM^∞ -converse consistent (DM^∞ -COCONS) if it satisfies the following condition: Let $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$ be an economy, and let $(x_i)_{i \in N}$ be an efficient allocation in \mathcal{E} . If for all $S \subseteq N, S \neq \emptyset, N, (x_i)_{i \in S} \in \phi(\mathcal{E}^{S,x})$, then $(x_i)_{i \in N} \in \phi(\mathcal{E})$.

The following abstract equilibrium notion will be useful in our next axiomatization.

Definition 5: Let $\mathcal{E} = \langle N, (X_i, \succeq_i, Y_i)_{i \in N} \rangle$ be an economy in E_1 . A W-equilibrium of \mathcal{E} is a collection $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ such that:

- (i) $(x_i)_{i \in N}$ is an allocation in \mathcal{E} ;
- (ii) $Y_i \subseteq C_i$;
- (iii) $\sum_{k \in S} C_k - \sum_{k \in S \setminus \{i\}} W_k(x_k) \subseteq C_i \quad \forall i \in S, \forall S \subseteq N$;
- (iv) $P_i(x_i) \cap C_i = \emptyset$

Note that an E-equilibrium and a W-equilibrium differ in the recontracting conditions on the feasible sets.

Serrano and Volij (1997) prove the following ⁸:

Theorem 5: *Let \mathcal{E} be an economy. An allocation $(x_i)_{i \in N} \in \mathcal{W}(\mathcal{E})$ if and only if it can be supported by a W-equilibrium. Moreover, the sets $(C_i)_{i \in N}$ corresponding to the W-equilibrium can be chosen to be:*

$$C_i = Z_i^{N,x}, i \in N. \tag{7}$$

Based on Theorem 5, we can prove our final result:

Theorem 6: *A solution ϕ on E_1 satisfies OPR, DM^∞ -CONS and DM^∞ -COCONS if and only if $\phi = \mathcal{W}$ (i.e., $\phi(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ for every economy $\mathcal{E} \in E_1$).*

Proof: First we show that the Walras solution satisfies OPR, DM^∞ -CONS and DM^∞ -COCONS. By definition, the Walras solution satisfies OPR. To see that the Walras solution is DM^∞ -consistent, let \mathcal{E} be an economy and let $(x_i)_{i \in N} \in \mathcal{W}(\mathcal{E})$. By Theorem 5, $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is a W-equilibrium in \mathcal{E} , where

$$C_i = Z_i^{N,x}, \tag{8}$$

Let $S \subseteq N$ be a coalition. We need to show that $(x_i)_{i \in S} \in \mathcal{W}(\mathcal{E}^{S,x})$. By Theorem 5, it is enough to show that $\langle (x_i)_{i \in S}, (C_i)_{i \in S} \rangle$ is a W-equilibrium.

To show (i) in the definition of W-equilibrium note that

$$\sum_{i \in S} x_i \in \left[Y_N - \sum_{k \in N \setminus S} W_k(x_k) \right].$$

Pick an agent $j \in S$. The last expression can be rearranged to get:

$$\sum_{i \in S} x_i \in \left[Y_{(N \setminus S) \cup \{j\}} - \sum_{k \in N \setminus S} W_k(x_k) \right] + Y_{S \setminus \{j\}},$$

which is included in

$$Z_j^{(N \setminus S) \cup \{j\}}(1) + \sum_{i \in S \setminus \{j\}} Z_i^{(N \setminus S) \cup \{i\}}(0) \subseteq \sum_{i \in S} Y_i^{S,x}.$$

Therefore, (i) holds. Notice that for all $i \in S$ we have

$$Y_i^{S,x} = \bigcup_{F \subseteq N \setminus S} Z_i^{F \cup \{i\},x} \subseteq \bigcup_{Q \subseteq N \setminus \{i\}} Z_i^{Q \cup \{i\},x} = C_i,$$

which means that (ii) is satisfied.

⁸ See also Theorem 1 in Dagan (1996) for the same result in his context of exchange economies.

Lastly, (iii) and (iv) are satisfied because $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is a W-equilibrium in \mathcal{E} . Therefore, the Walras solution satisfies DM^∞ -CONS.

Next we show that the Walras solution satisfies DM^∞ -COCONS. Let $(x_i)_{i \in N}$ be an efficient allocation in \mathcal{E} . Assume that for all $S \subseteq N, S \neq \emptyset, N, (x_i)_{i \in S} \in \mathcal{W}(\mathcal{E}^{S,x})$. We must show that $(x_i)_{i \in N} \in \mathcal{W}(\mathcal{E})$. By Theorem 5, it is enough to show that $\langle (x_i)_{i \in N}, (C_i)_{i \in N} \rangle$ is a W-equilibrium for C_i defined in Eq. (8).

Condition (i) is satisfied by assumption. By definition of C_i , conditions (ii) and (iii) are immediately satisfied as well. By assumption and by Theorem 5, for all $S \subseteq N, i \in S, \langle (x_i)_{i \in S}, (C_i^{S,x})_{i \in S} \rangle$ is a W-equilibrium for $\mathcal{E}^{S,x}$. Hence, it must satisfy (iv), that is, $P_i(x_i) \cap C_i^{S,x} = \emptyset$. In particular, for $S = \{i\}, P_i(x_i) \cap C_i^{\{i\},x} = \emptyset, \forall i \in N$. But notice that, by definition of $C_i^{\{i\},x} = C_i$. Therefore, as claimed, the Walras solution satisfies DM^∞ -COCONS.

To prove the Theorem, we need to show the opposite implication. This follows from the following two lemmas.

Lemma 8: *Let ϕ be a solution on E_1 that satisfies OPR and DM^∞ -CONS. For all $\mathcal{E} \in E_1$, if $x \in \phi(\mathcal{E})$, then x is an efficient allocation in \mathcal{E} .*

Proof: It is analogous to that of Lemma 1 and hence is left to the reader. □

Lemma 9: *Let φ be a DM^∞ -consistent and OPR solution on E_1 and let ψ be DM^∞ -converse consistent on the same class. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies \mathcal{E} , then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all economies \mathcal{E} in E_1 .*

Proof: It is analogous to that of Lemma 2 and is left to the reader. □

Let now ϕ be a solution on E_1 that satisfies OPR, DM^∞ -CONS and DM^∞ -COCONS. Since both ϕ and \mathcal{W} satisfy OPR, they coincide for all one-person economies. Further, since both satisfy DM^∞ -CONS and DM^∞ -COCONS, by Lemma 9, they must coincide for all economies. □

To demonstrate that the axioms used in the characterization are independent, consider the following examples.

Example 11: The empty solution satisfies DM^∞ -CONS and DM^∞ -COCONS, but does not satisfy OPR.

Example 12: Let ϕ be a solution on the class of all additive economies defined as follows:

$$\phi(\mathcal{E}) = \begin{cases} \mathcal{RH}(\mathcal{E}) & \text{if } \mathcal{E} \text{ is a one - person economy} \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ satisfies OPR and DM^∞ -CONS. By Theorem 6, it cannot satisfy DM^∞ -COCONS because $\phi \neq \mathcal{W}$.

Example 13: It is easy to see that $\mathcal{P}\mathcal{O}$ satisfies OPR and DM^∞ -COCONS. By Theorem 4, it cannot satisfy DM^∞ -CONS because $\mathcal{P}\mathcal{O} \neq \mathcal{W}$.

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