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Decentralized information and the Walrasian outcome: a pairwise meetings market with private values

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Abstract

I study a one-time entry market for a single indivisible good, where buyers and sellers, privately informed regarding their valuation for the good, are randomly matched, bargain, and in the event of agreement, trade and exit the market. Each agent's search procedure is modeled as a sequence of discrete double auctions. For each value of the discount factor, the equilibrium behavior of traders satisfies a certain property of monotonicity within each side of the market—the lower a trader's potential surplus the tougher his market position. As discounting is removed, equilibria with Walrasian and non-Walrasian features persist, although sufficient conditions are identified to single out the Walrasian outcome.

JEL classification: C72; C78; D82; D83

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1. Introduction

Consider a market for an indivisible good. There is a continuum of buyers with different valuations for the good and a continuum of sellers who can produce it at different unit costs. It is realistic to assume that these values (buyers' valuations and sellers' costs) are private information to each trader. Suppose also that knowledge of their own value does not give them additional information about the values of each of the others. This is a problem of independent private values asymmetric information, where, although the overall

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distributions of valuations and costs may be common knowledge among traders, they do not know each other's values when they meet to trade.

The question of how decentralized information of this kind may interfere with market performance was first raised by Hayek (1945), Arrow (1959) and Hurwicz (1973), among others. The issue is an important one: when being asked to report it, traders may strategically lie about their private information, thereby distorting the true demand and supply curves and creating inefficiencies. The standard market solution to the problem, also for this setup, is the competitive or Walrasian equilibrium, which succeeds yielding the efficient outcome. However, this mechanism does not explain how prices are formed to overcome the problem caused by private information.

The issue analyzed here is the presence of private values uncertainty in a decentralized market. The paper studies a specific mechanism of trade, where prices are endogenously determined by the actions of traders; to the best of our knowledge, this is the first study where each agent faces an infinite sequence of bilateral double auctions. In the context of models with pairwise meetings, Gale (1987) was the first to ask questions related to the ones here.[1] He studies various versions of a pairwise meetings procedure in a market for an indivisible good and shows that in the limit as the discount factor δ goes to 1, all equilibria yield the Walrasian outcome. His procedure is based on a sequential bargaining game when a pair meets. The analysis is carried out assuming complete information, but the conclusions extend to the case of private values uncertainty.[2]

On the other hand, we have the pioneering work of Wolinsky (1990) (W in the sequel) in studying a pairwise meetings market with common values asymmetric information.[3] W and Blouin and Serrano (2001) (BS in the sequel) establish that, for the case of common values uncertainty, there seems to be little room for a positive result of information revelation and efficiency of decentralized markets based on bilateral trade. In these papers, as frictions are removed (as $\delta \to 1$), the quality of matches deteriorates: the informed agents leave the market early and the equilibrium outcome is driven mainly by noise. In addition, BS find fear-based equilibria with a small volume of trade; pessimistic beliefs lead uninformed agents to refuse to trade and market activity suffers as a consequence. Overall, little support is found for centralized equilibrium paradigms, such as the rational expectations equilibrium.

The pertinent methodological question of what accounts for these differences is the one asked here. The models described differ in three aspects. (1) The information asymmetries are of the private values type in Gale (1987) and of the common values variety in W–BS. (2) The trading procedure in each meeting in the Gale model is sequential, allowing perfection arguments to refine the set of equilibria, while it is a simultaneous stage game in W–BS. (3)

[1] Samuelson (1992) is the first model involving pairwise meetings and private values: his concern is to establish that delays may happen in equilibrium. See also Serrano and Yosha (1996b), McLennan and Sonnenschein (1991), discussed in Dagan et al. (1998, 2000), and Moreno and Wooders (2002). Osborne and Rubinstein (1990, Chapters 6–10) and Gale (2000) provide lucid presentations of models with pairwise meetings.

[2] Exactly the same happens in Gale (1986a). It is far more subtle, though, to see how the arguments in Gale (1986b) would extend to incomplete information, given the difficulty of writing down equilibrium strategies in this case. In general, existence of equilibrium in these models is not a trivial matter. The current paper deals with this issue at length.

[3] Serrano and Yosha (1993, 1996a) study different aspects of W's model, and Blouin and Serrano (2001) relax its strong steady-state assumption.

While prices are "fully endogenous" in Gale's procedure (the proposer in the bargaining round is free to propose any price), the set of possible prices is exogenously fixed in W–BS. One could argue that the modeling choices in W–BS, while driven by tractability considerations, are nonetheless additional frictions imposed on the model. The present paper sheds light on this debate by studying for the private values case a procedure that generalizes that in W–BS, retaining its features (2) and (3). As in BS, I shall not rely on a steady-state assumption in analyzing the procedure for the private values case, but its properties in this context turn out to be much more promising. The model will always deliver a class of equilibria with Walrasian features. Thus, the additional frictions (2) and (3) in the W–BS procedure are compatible with efficient equilibria in the private values case. In addition, inefficient equilibria and the forces behind them are identified.

In specifying the trading procedure, I face the usual tradeoff between tractability and realism. I enrich W's elegant model, also used in BS, by introducing more prices at which trade may take place. Yet tractability restrictions cannot be ignored and the set of prices allowed will be finite. A description of the model follows.

Fixed populations of buyers and sellers are present at the outset, and no other agents will enter this market. Each agent is matched in each period to an agent of the opposite type. When matched, traders bargain. If they agree on a price, they transact and leave the market; otherwise, they remain in it to be matched again. Matches are random and anonymous.

As for the bargaining technology, each trader plays a sequence of *discrete* double auctions, one in each meeting. For simplicity of exposition, I write down the model, where (a) only three positions can be adopted by a trader upon meeting another (tough, soft or very soft); (b) at most one switch in these positions over time is allowed for each trader; and (c) the middle position is associated with announcing the Walrasian price, p^{W}.[4]

The "positions" are modeled as *simultaneous* announcements within each meeting. A tough position means for a seller to ask for the highest price (p^{H}) and for a buyer to offer the lowest price (p^{L}). At the other extreme, a very soft position means for a seller to ask for p^{L} and for a buyer to offer p^{H}. The middle ground is the soft position, where a seller can ask for and a buyer can offer an intermediate price, say p^{W}. Trade in a meeting takes place if and only if the buyer's offered price is at least as high as the seller's asked price. If trade occurs, the transaction is closed at the average of the two prices. An agent may play tough forever, in which case he would only trade upon meeting a very soft agent of the opposite side of the market. An agent playing soft forever will trade only with soft or very soft agents, while an agent playing very soft will trade immediately, no matter his opponent's position. In addition, an agent can start exploring the market by playing tough and switch to either a soft or very soft position to increase his chances of trade. This structure includes a rich enough set of prices (a total of six, one of which is the Walrasian one), but it lends itself to the analysis.

There is discounting across periods, represented by $\delta < 1$, common to all agents. As usual, the focus shall be primarily on the performance of the trading procedure when δ is close to 1.

[4] None of the results depend on these simplifying assumptions. See Remark 3 in the last section for a discussion of the more general model, where a large finite set of prices is allowed and multiple changes in positions is permitted, always from tougher to softer.

For a fixed δ, we find that the equilibrium behavior of agents can be neatly classified.[5] Intramarginal traders, those found in the demand and supply curves beyond the competitive output, always play tough. Marginal traders, those with values equal to the Walrasian price, either always play tough or switch to soft at some point. Low surplus traders (buyers with valuation in the interval $(p^{\mathrm{W}}, p^{\mathrm{H}})$ and sellers with cost in $(p^{\mathrm{L}}, p^{\mathrm{W}})$) always find it optimal to adopt the soft position at some point, but never a very soft one. Finally, high surplus traders (buyers with valuations exceeding p^{H} and sellers with cost lower than p^{L}) stand more to lose by not trading, and may adopt in equilibrium either a soft or a very soft position after having possibly gone through a tough phase.

As $\delta \to 1$, the model retains equilibria yielding Walrasian and non-Walrasian outcomes. In the equilibria supporting the Walrasian outcome, the high surplus traders do not use the switch to a very soft position. Instead, they adopt a soft position after having possibly played a tough phase. Consequently, all trade takes place at p^{W}. In addition, just enough marginal traders to clear the market use a soft position, while the rest always play tough (recall that these traders are indifferent between trading at p^{W} and not trading at all). But precisely if these two types of traders choose to behave differently, non-Walrasian outcomes result in equilibrium.

Accordingly, one can identify the forces that may prevent the Walrasian outcome. (1) If there is a sufficiently high mass of marginal traders adopting a soft position and therefore willing to trade, they may end up rationing other traders with positive surplus on the same side of the market, who will be prevented from trading. (2) The high surplus traders may fear going home without trading. This is possible if they never switch to very soft, given the presence of intramarginal traders, who always play tough. Correspondingly, they may in equilibrium choose to play very soft at some point, yielding non-Walrasian outcomes where trade takes place at multiple prices and intramarginal agents get to trade.[6] (3) In addition, due to the simultaneity of the bargaining game, one can identify other coordination failures. In them, the competitive output is traded at the Walrasian price and no intramarginal agent transacts, but delay elapses until trade begins. Efficiency may suffer as a result.

Having identified the effects that prevent the Walrasian price from occuring, one can state sufficient conditions to eliminate them. On the one hand, the rationing created by marginal traders disappears if their mass is negligible (for example, if the number of types goes to infinity).[7] On the other hand, one needs to rule out high surplus traders. Since these are defined as the types that lie outside of the range of prices allowed by the model, they disappear in situations where the highest price considered is the highest consumer valuation and the lowest is the lowest cost. In sum, in markets approximating continuous demand and supply curves where the range of prices in the procedure covers the entire range of values,

[5] See Moreno and Wooders (2002) for related results in a model with one type of sellers and two types of buyers (the trade patterns over time, and not so much the convergence question, is the main focus of their paper). The same effect is found in the literature on bilateral bargaining with incomplete information (see Osborne and Rubinstein (1990, Chapter 5) and the references therein).

[6] See Chamberlin (1948), an early experimental paper, who reports similar phenomena in his experiment involving pairwise meetings. See also Holt (1995, pp. 368–374) for a survey of experimental results on double auctions, where the finding tends to be a strong support of the Walrasian outcome.

[7] A slight change in the model, as in Gale (1987), so that the agents' entry decisions are endogenous and require a small positive entry fee, would also eliminate this effect.

the outcome of this decentralized trade mechanism yields asymptotically the Walrasian price and output exchanged. But it may or may not approach efficiency because of the equilibria where the Walrasian output is only exchanged after considerable delay.

The approach in Gale (1987) and the one in the current paper are quite distinct. Gale's arguments cleverly rely on the study of each agent's value function. In the current study, a useful characterization of traders' equilibrium strategies is uncovered and new equilibria appear. Nonetheless, after removing the finite sets of traders' types and of allowed prices, the present model confirms Gale's one-price result and has a strong Walrasian flavor.

This paper is also related to the literature on optimal mechanisms and double auctions when the number of traders is large (see for, e.g. Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), and Rustichini et al. (1994)). This literature concentrates on the symmetric equilibria of their trading procedures and finds asymptotic convergence to efficiency. The price is not chosen by the auctioneer to clear the market, but it is *centrally* calculated by the entire strategy profile. In contrast, the current paper studies the set of all equilibria by adopting the distributional approach, goes one step beyond in the *decentralization* of prices and emphasizes the dynamics of trade over time.

Here is the plan of the paper. Section 2 gives the specifics of the model. The equilibrium notion is found in Section 3. The important problem of its existence is treated in Section 4 (which readers less familiar with technical arguments can skip without loss of continuity). Section 5 contains the characterization of equilibrium behavior. Section 6 is a collection of examples of equilibria. The asymptotic result can be found in Section 7, and Section 8 concludes.

2. The model

There are two populations of agents in the economy: sellers and buyers. There is a single indivisible good. Each seller has one unit of the good for sale, and each buyer is interested in buying one unit. Each population is an atomless continuum of measure 1.[8]

Time elapses discretely according to $t = 0, 1, 2, \dots$. All agents enter the market at the beginning of period 0. There is no entry of new agents in subsequent periods.

In period 0, each agent is randomly matched with one agent of the other population. The pair then tries to agree on a price at which to transact the good. If there is agreement, the transaction takes place, the two agents receive their payoffs and exit the market forever. If there is disagreement, no transaction takes place and the two agents remain in the market. In period 1, each of the *remaining* agents is again randomly matched with an agent of the other population. The pair tries to agree on a price, etc. This cycle is repeated infinitely many times, or until all agents have transacted and left the market. Note that there is always an equal measure of sellers and buyers remaining in the market.

The payoffs to two agents reaching agreement depend on the price at which they transact and on their own valuation for the good. Therefore, if a buyer of valuation u_i and a seller of cost c_j transact at price p, the buyer obtains an instantaneous payoff of $u_i - p$, whereas

[8] We are interested in analyzing a model where the matching friction is small. In our model, it is zero: a trader gets matched every period that he remains in the market. Models with small but positive matching friction give a similar prediction to ours, as the matching friction vanishes.

the seller's instantaneous payoff is $p - c_j$. The valuations u_i and c_j are private information to the two traders.

A fraction α_i of the initial population of buyers has valuation u_i for the good, where $i = 1, 2, \ldots, I$ and $u_1 > u_2 > \cdots > u_I$. Similarly, a fraction β_j of the initial population of sellers has cost c_j, where $j = 1, 2, \ldots, J$ and $c_1 < c_2 < \cdots < c_J$. We assume that $\alpha_i > 0$ for all i, and that $\beta_j > 0$ for all j. We shall assume that $u_1 > c_1$, and that there exists i and j such that $u_i < c_j$. These assumptions ensure that, at the competitive equilibrium, some but not all units are traded: this is the most general case. We shall denote the competitive or Walrasian price by p^{W}. Finally, we shall be ultimately concerned with the model where I and J are arbitrarily large integers, in an attempt to approximate continuous demand and supply schedules.

Traders are immersed in a game that looks like a "war of attrition." Each of them will start by playing "tough" to the market, which will entail trade at a very advantageous price or disagreement. After the "tough" phase, a trader can switch to either a "soft" phase or a "very soft" one. A very soft position guarantees the trader to leave the market immediately, trading at much less favorable prices. A soft position increases (with respect to the tough phase) the trader's probability to trade immediately, but this is not unity. Thus, a switch to a soft position is a middle ground and actual trade may still take a while, even after the trader has switched to soft. To simplify, the number of switches in each trader's position is at most one: a trader can switch from "tough" to "soft," or from "tough" to "very soft." Any other switch is ruled out. Of course, a trader can choose to never switch positions, either because he always plays "tough," "soft" or "very soft."

Thus, in each period bargaining between a seller and buyer proceeds as follows. The two agents simultaneously take bargaining positions in a discrete double auction. Each of them announces a price from the set $\{p^{\mathrm{L}}, p^{\mathrm{W}}, p^{\mathrm{H}}\}$. Trade takes place if and only if the price announced by the buyer is at least as high as that asked by the seller. The price at which the transaction takes place is the average of the two prices announced. We shall say that the buyer plays *tough* if he announces p^{L}, *soft* if he announces p^{W} and *very soft* if he announces p^{H}. Similarly, a seller plays *tough* if he asks for p^{H}, *soft* if he asks for p^{W} and *very soft* if he announces p^{L}. Denote by p^{FG}, the average of p^{F} and p^{G} for $\mathrm{F}, \mathrm{G} \in \{\mathrm{L}, \mathrm{W}, \mathrm{H}\}$. We assume that

$$0 \le c_1 \le p^{\mathrm{L}} < u_I \le p^{\mathrm{W}} \le c_J < p^{\mathrm{H}} \le u_1. \tag{1}$$

No specific assumptions are required for p^{LW}, p^{LH} or p^{WH}. To understand the alternatives offered to a trader in these rules, consider a buyer of valuation u_i. He can play very soft from the outset. In this case, he will announce p^{H} and exit the market immediately, trading at one of three possible prices: p^{H} if he meets a tough seller, p^{WH} if he meets a soft seller, or p^{LH} if he meets a very soft seller. Alternatively, this buyer can play soft from the outset by announcing p^{W}. Then, in every period, he will leave the market if he meets a soft seller (trading at p^{W}) or a very soft seller (trading at p^{LW}), but he will disagree and be around the next period when he meets a tough seller. Finally, this buyer can start by playing tough (announcing p^{L}). For as long as he holds the tough position, he guarantees that trade will take place only at p^{L} if he meets a very soft seller. If he encounters a soft or tough seller, disagreement arises. A buyer can either play always tough, or eventually switch to either soft or very soft at some point. The reader is invited to run through similar considerations for a typical seller.

There is discounting. We shall assume that all traders have a common per period discount factor δ, $0 < \delta < 1$. Perpetual disagreement entails a payoff of zero.

We turn now to individual agents' strategies. There are two things to consider. The first is anonymity. An agent never knows the identity of his opponent and can never tell his opponent's valuation or cost. He must treat all opponents the same way. Second, we assume that an agent observes only his own history, but he is ignorant about the other agents' histories. An agent can have two types of histories: either he has played tough until now (and met a sequence of tough and soft agents), or he switched to soft a number of periods ago (and since then has only met tough agents). In any event, since personal histories are private, equilibria in this model will have to be history independent.

It follows from these considerations that an agent's equivalence class of strategies for the game can be represented by a non-negative integer and one of two possible stopping rules (enhancing this set by the singleton $\{\infty\}$). That is, a strategy will specify the number of periods he is prepared to play tough, along with the final decision to switch to either soft or very soft. He can therefore calculate right at the start of the game how long it will be optimal to play tough and when it will be optimal to switch to either soft or very soft. In other words, he can decide from the outset how long to hold out for the most advantageous price, and when and how to give in. Thus the strategy space is $A \equiv \mathbb{N}_S \cup \mathbb{N}_{VS} \cup \{\infty\}$, where $t_S \in \mathbb{N}_S = \{0_S, 1_S, 2_S, \dots\}$ means to play tough for t periods and switch to soft in period t. Similarly, $t_{VS} \in \mathbb{N}_{VS} = \{0_{VS}, 1_{VS}, 2_{VS}, \dots\}$ means to play tough for t periods and switch to very soft in period t. An agent playing ∞ plays tough all the time.

It is easy to see that any Nash equilibrium outcome in our model can be supported by a perfect Bayesian equilibrium, where beliefs are as follows: each agent believes on and off the equilibrium path that a full measure of agents continues to play the given equilibrium (see, for example, Osborne and Rubinstein (1990, Chapter 8)). Therefore, we seek a Nash equilibrium, a profile of strategies where each agent is maximizing his expected payoff, given the strategies of the other agents. All parameters are common knowledge, as are all equilibrium strategies.

We shall denote the game described in this section by Γ.

3. Equilibrium, expected payoffs and trade statistics

An agent will in principle act differently as a function of his valuation or cost. There are thus $I + J$ types of behavior to account for: buyers of valuation u_i, $i = 1, \dots, I$ and sellers of cost c_j, $j = 1, \dots, J$. Let K be the set of $I + J$ possible types of agents. We shall denote an arbitrary type of an agent by the letter k. Similarly, type i will denote a buyer with valuation u_i, while type j will denote a seller of cost c_j.

Agents belonging to the same type solve the same problem. However, they will not necessarily adopt the same strategy, as several strategies may be optimal. For any subset of possible strategies $X \subset A$, we denote by $\phi^k(X)$ the fraction of the initial population of type-k agents who in equilibrium play strategies contained in X. For simplicity we write $\phi^k(t_S) \equiv \phi^k(\{t_S\})$ and $\phi^k([a_S, b_S]) \equiv \phi^k(\{a_S, \dots, b_S\})$, for all $a_S, b_S \in \mathbb{N}_S$. Also,

$\phi^k(t_{\mathrm{VS}}) \equiv \phi^k(\{t_{\mathrm{VS}}\})$ and $\phi^k([a_{\mathrm{VS}}, b_{\mathrm{VS}}]) \equiv \phi^k(\{a_{\mathrm{VS}}, \ldots, b_{\mathrm{VS}}\})$, for all $a_{\mathrm{VS}}, b_{\mathrm{VS}} \in \mathbb{N}_{\mathrm{VS}}$. We shall use the shorthand $\phi^{\mathrm{B}}(a)$ to denote the total fraction of the initial set of buyers who play strategy a, i.e. $\phi^{\mathrm{B}}(a) \equiv \sum_{i=1}^{I} \alpha_i \phi^i(a)$. Similarly, we use $\phi^{\mathrm{S}}(a)$ for the sellers side: $\phi^{\mathrm{S}}(a) \equiv \sum_{j=1}^{J} \beta_j \phi^j(a)$.

The set function ϕ^k has all the features of a probability measure, and we will treat it as such. The set of probability measures on A will be denoted Φ. Candidates for equilibrium will take the form $\phi = (\phi^k)_{k\in K} \in \Phi^{I+J}$.

The function $\pi^k(a;\phi)$ will be the expected payoff to a type-k agent from playing strategy $a \in A$ (i.e. playing tough a times and playing either soft in period a if $a = a_{\mathrm{S}}$, or very soft in period a if $a = a_{\mathrm{VS}}$) or never switching if $a = \infty$, given that other agents' strategies conform to ϕ. Often we will omit strategies from the argument and write simply $\pi^k(a)$.

Definition 1. A Nash equilibrium of the game Γ consists of $\phi = (\phi^k)_{k\in K} \in \Phi^{I+J}$ such that, for all $a \in A$ and for all $k \in K$:

$$\phi^k(a) > 0 \quad \text{implies} \quad a \in \arg\max_{b\in A} \pi^k(b;\phi). \tag{2}$$

We now calculate expected payoffs for each type of agent. Agents know the distributions of equilibrium strategies for the types they are likely to be paired with, and use these to calculate the probability of meeting a tough, soft or very soft opponent in any given period.

Define σ_t^{VS} to be the proportion of sellers active in period t who play very soft in that period. Let σ_t^{S} be the proportion of sellers active in period t who play soft in that period. Of course, a fraction $1-\sigma_t^{\mathrm{VS}}-\sigma_t^{\mathrm{S}}$ of the sellers active in period t play tough in that period. The fractions ρ_t^{VS}, ρ_t^{S} and $1-\rho_t^{\mathrm{VS}}-\rho_t^{\mathrm{S}}$ are similarly defined for the active buyers in period t.

For example, $\sigma_0^{\mathrm{VS}} = \phi^{\mathrm{S}}(0_{\mathrm{VS}})$, $\sigma_0^{\mathrm{S}} = \phi^{\mathrm{S}}(0_{\mathrm{S}})$, $\rho_0^{\mathrm{VS}} = \phi^{\mathrm{B}}(0_{\mathrm{VS}})$, and $\rho_0^{\mathrm{S}} = \phi^{\mathrm{B}}(0_{\mathrm{S}})$.

To calculate the same proportions for periods $t > 0$, one has to take into account the size of the market in that period and the traders' probability of having traded prior to that date. Recall in this respect that the proportions ϕ^k are defined over the initial population, whereas σ_t^{VS}, σ_t^{S}, ρ_t^{VS} and ρ_t^{S} are proportions of the active traders in period t. It can be checked that the size of the market in period t, i.e. the mass of traders remaining in period t in each of the two sides of the market is:

$$\begin{aligned} N_t = {} & [\phi^{\mathrm{B}}(t'_{\mathrm{VS}} \ge t_{\mathrm{VS}}) + \phi^{\mathrm{B}}(t'_{\mathrm{S}} \ge t_{\mathrm{S}}) + \phi^{\mathrm{B}}(\infty)] \\ & \times [\phi^{\mathrm{S}}(t'_{\mathrm{VS}} \ge t_{\mathrm{VS}}) + \phi^{\mathrm{S}}(t'_{\mathrm{S}} \ge t_{\mathrm{S}}) + \phi^{\mathrm{S}}(\infty)] \\ & + \phi^{\mathrm{B}}([0_{\mathrm{S}}, (t-1)_{\mathrm{S}}])[\phi^{\mathrm{S}}(t'_{\mathrm{VS}} \ge t_{\mathrm{VS}}) + \phi^{\mathrm{S}}(t'_{\mathrm{S}} \ge t_{\mathrm{S}}) \\ & + \phi^{\mathrm{S}}(\infty)] + \phi^{\mathrm{S}}([0_{\mathrm{S}}, (t-1)_{\mathrm{S}}])[\phi^{\mathrm{B}}(t'_{\mathrm{VS}} \ge t_{\mathrm{VS}}) + \phi^{\mathrm{B}}(t'_{\mathrm{S}} \ge t_{\mathrm{S}}) + \phi^{\mathrm{B}}(\infty)]. \end{aligned}$$

Using this, we can write as an example

$$\sigma_1^{\mathrm{VS}} = \frac{\phi^{\mathrm{S}}(1_{\mathrm{VS}})[1-\phi^{\mathrm{B}}(0_{\mathrm{VS}})]}{N_1},$$

$$\sigma_1^{\mathrm{S}} = \frac{\phi^{\mathrm{S}}(1_{\mathrm{S}})[1-\phi^{\mathrm{B}}(0_{\mathrm{VS}})] + \phi^{\mathrm{S}}(0_{\mathrm{S}})[1-\phi^{\mathrm{B}}(0_{\mathrm{VS}})-\phi^{\mathrm{B}}(0_{\mathrm{S}})]}{N_1},$$

and so on. In the earlier expressions, the numerators detail the mass of sellers playing "very soft" or "soft" in period 1. Fortunately, much of the analysis will be executed with little need to carry over these messy expressions.

Next we proceed to write the expected payoffs for the different traders. Consider first a buyer of valuation u_i. His expected payoff to playing strategy a_{VS} (playing "tough" in all periods $t < a$ and switching to "very soft" in period a), a_{S} (playing "tough" in all periods $t < a$ and switching to "soft" in period a), or ∞ (always playing "tough") is:[9]

$$\pi^i(a_{\mathrm{VS}}) = \sum_{t=0}^{a-1} \delta^t \sigma_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\sigma_r^{\mathrm{VS}})(u_i - p^{\mathrm{L}}) + \delta^a \Pi_{r=0}^{a-1}(1-\sigma_r^{\mathrm{VS}}) \times [\sigma_a^{\mathrm{VS}}(u_i - p^{\mathrm{LH}}) + \sigma_a^{\mathrm{S}}(u_i - p^{\mathrm{WH}}) + (1-\sigma_a^{\mathrm{VS}} - \sigma_a^{\mathrm{S}})(u_i - p^{\mathrm{H}})],$$

$$\pi^i(a_{\mathrm{S}}) = \sum_{t=0}^{a-1} \delta^t \sigma_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\sigma_r^{\mathrm{VS}})(u_i - p^{\mathrm{L}}) + \Pi_{r=0}^{a-1}(1-\sigma_r^{\mathrm{VS}}) \times \left[\sum_{t=a}^{\infty} \Pi_{r=a}^{t-1}(1-\sigma_r^{\mathrm{VS}} - \sigma_r^{\mathrm{S}})\delta^t(\sigma_t^{\mathrm{VS}}(u_i - p^{\mathrm{LW}}) + \sigma_t^{\mathrm{S}}(u_i - p^{\mathrm{W}}))\right],$$

$$\pi^i(\infty) = \sum_{t=0}^{\infty} \delta^t \sigma_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\sigma_r^{\mathrm{VS}})(u_i - p^{\mathrm{L}}).$$

Similarly, for a seller of cost c_j, his expected payoff to playing strategies a_{VS}, a_{S} or ∞ is:

$$\pi^j(a_{\mathrm{VS}}) = \sum_{t=0}^{a-1} \delta^t \rho_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\rho_r^{\mathrm{VS}})(p^{\mathrm{H}} - c_j) + \delta^a \Pi_{r=0}^{a-1}(1-\rho_r^{\mathrm{VS}})[\rho_a^{\mathrm{VS}}(p^{\mathrm{LH}} - c_j) + \rho_a^{\mathrm{S}}(p^{\mathrm{LW}} - c_j) + (1-\rho_a^{\mathrm{VS}} - \rho_a^{\mathrm{S}})(p^{\mathrm{L}} - c_j)],$$

$$\pi^j(a_{\mathrm{S}}) = \sum_{t=0}^{a-1} \delta^t \rho_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\rho_r^{\mathrm{VS}})(p^{\mathrm{H}} - c_j) + \Pi_{r=0}^{a-1}(1-\rho_r^{\mathrm{VS}}) \times \left[\sum_{t=a}^{\infty} \Pi_{r=a}^{t-1}(1-\rho_r^{\mathrm{VS}} - \rho_r^{\mathrm{S}})\delta^t(\rho_t^{\mathrm{VS}}(p^{\mathrm{WH}} - c_j) + \rho_t^{\mathrm{S}}(p^{\mathrm{W}} - c_j))\right],$$

$$\pi^j(\infty) = \sum_{t=0}^{\infty} \delta^t \rho_t^{\mathrm{VS}} \Pi_{r=0}^{t-1}(1-\rho_r^{\mathrm{VS}})(p^{\mathrm{H}} - c_j).$$

We can also determine the profile of trade. In each period $t \in \mathbb{N}$, $\sigma_t^{\mathrm{VS}}\rho_t^{\mathrm{VS}}N_t$ units are traded at price p^{LH}, $\sigma_t^{\mathrm{VS}}\rho_t^{\mathrm{S}}N_t$ units at p^{LW}, $\sigma_t^{\mathrm{VS}}(1-\rho_t^{\mathrm{VS}}-\rho_t^{\mathrm{S}})N_t$ units at p^{L}, $\sigma_t^{\mathrm{S}}\rho_t^{\mathrm{VS}}N_t$ units are traded at price p^{WH}, $(1-\sigma_t^{\mathrm{VS}}-\sigma_t^{\mathrm{S}})\rho_t^{\mathrm{VS}}N_t$ units at p^{H}, and $\sigma_t^{\mathrm{S}}\rho_t^{\mathrm{S}}N_t$ units at p^{W}. The rest of units are not traded in period t.

[9] Throughout the paper, I adopt the conventions that $\sum_{t=0}^{-1}(\cdot) = 0$ and $\Pi_{t=0}^{-1}(\cdot) = 1$.

Therefore, the following statistic defines the volume of units traded:

$$V = \sum_{t \in \mathbb{N}} \{\sigma_t^{\text{VS}} + \rho_t^{\text{VS}}[1 - \sigma_t^{\text{VS}}] + \sigma_t^{\text{S}} \rho_t^{\text{S}}\} N_t.$$

We shall also be concerned with the following ratio, which, in a given equilibrium, measures the proportion of trade that takes place at the Walrasian price p^{W}:

$$V_{\text{W}} = \frac{\sum_{t \in \mathbb{N}} \sigma_t^{\text{S}} \rho_t^{\text{S}} N_t}{V}.$$

To fix ideas, we shall concentrate most of the time on a market for which there is a unique competitive price, $p^{\text{W}} = u_i$, for some i (this is one of the two generic cases, and it is without loss of generality).[10] Denote by Q^{W} the mass of units exchanged at the competitive equilibrium. We shall say that *the market yields the Walrasian outcome* whenever:

(i) $V_{\text{W}} = 1$;
(ii) $V = Q^{\text{W}}$;
(iii) sellers with cost c_j trade if and only if $c_j < p^{\text{W}}$. Buyers with valuation u_i trade if $u_i > p^{\text{W}}$ and do not trade if $u_i < p^{\text{W}}$. As for those buyers with valuation $u_i = p^{\text{W}}$, just enough of them trade so that the total mass of trade is Q^{W};
(iv) sellers with cost c_j who trade obtain an expected payoff of $p^{\text{W}} - c_j$, and buyers of valuation u_i who trade obtain an expected utility of $u_i - p^{\text{W}}$.

4. Existence of equilibrium

In this section we establish existence of a Nash equilibrium in the game Γ for any value of the discount factor $\delta \in (0, 1)$. The arguments adapt results from the literature on existence in anonymous games and follow closely those in Blouin and Serrano (2001).

Proposition 1. *There exists a Nash equilibrium $\phi \in \Phi^{I+J}$ of the game Γ.*

Proof. The proof is a variation of Blouin and Serrano (2001), which follows Mas-Colell (1984), itself a reformulation of Schmeidler (1973).

Recall that $A \equiv \mathbb{N}_{\text{VS}} \cup \mathbb{N}_{\text{S}} \cup \{\infty\}$. We define the following metric d on A:

$$d(x_{\text{VS}}, y_{\text{VS}}) = d(y_{\text{VS}}, x_{\text{VS}}) = d(x_{\text{S}}, y_{\text{S}}) = d(y_{\text{S}}, x_{\text{S}}) = \frac{|x - y|}{(1+x)(1+y)} \quad \text{for}$$

$$x_{\text{VS}}, y_{\text{VS}} \in \mathbb{N}_{\text{VS}}, \quad x_{\text{S}}, y_{\text{S}} \in \mathbb{N}_{\text{S}};$$

$$d(x_{\text{VS}}, y_{\text{S}}) = d(y_{\text{S}}, x_{\text{VS}}) = d(x_{\text{S}}, y_{\text{VS}}) = d(y_{\text{VS}}, x_{\text{S}}) = \frac{1}{1+x} + \frac{1}{1+y} \quad \text{for}$$

$$x_{\text{VS}}, y_{\text{VS}} \in \mathbb{N}_{\text{VS}}, \quad x_{\text{S}}, y_{\text{S}} \in \mathbb{N}_{\text{S}};$$

[10] The other generic case is when the unique competitive price is $p^{\text{W}} = c_j$ for some j. The non-generic case happens when the set of equilibrium prices is the interval $[u_i, c_j]$ for some i and j.

$$d(x_{\mathrm{VS}},\infty)=d(\infty,x_{\mathrm{VS}})=d(x_{\mathrm{S}},\infty)=d(\infty,x_{\mathrm{S}})=\frac{1}{1+x}\quad\text{for}$$
$$x_{\mathrm{VS}}\in\mathbb{N}_{\mathrm{VS}},\quad x_{\mathrm{S}}\in\mathbb{N}_{\mathrm{S}};$$
$$d(\infty,\infty)=0.$$

This metric generates the following topology $\mathcal{T}$ for A.

$$\mathcal{T}\equiv\{X\subset A\,|\,\text{if }\infty\in X,\text{ then } \{t_{\mathrm{VS}},(t+1)_{\mathrm{VS}},\ldots\}\subset X\quad\text{for some } t_{\mathrm{VS}}\in\mathbb{N}_{\mathrm{VS}}\text{ and } \{t'_{\mathrm{S}},(t'+1)_{\mathrm{S}},\ldots\}\subset X\quad\text{for some } t'_{\mathrm{S}}\in\mathbb{N}_{\mathrm{S}}\}.$$

Note that all subsets of $\mathbb{N}_{\mathrm{VS}}\cup\mathbb{N}_{\mathrm{S}}$ are elements of $\mathcal{T}$. We assume throughout that A is endowed with this topology. Since every subset of A is either open or closed, the Borel sigma–algebra of A simply consists of all subsets of A. Since A is a metric space, it is normal.

Claim 1. *A is compact.*

Proof. Consider any open cover of A. One of the sets in the cover must include ∞ and therefore must also include the set $\{t_{\mathrm{VS}},(t+1)_{\mathrm{VS}},\ldots\}$ for some $t_{\mathrm{VS}}\in\mathbb{N}_{\mathrm{VS}}$ and the set of points $\{t'_{\mathrm{S}},(t'+1)_{\mathrm{S}},\ldots\}$ for some $t'_{\mathrm{S}}\in\mathbb{N}_{\mathrm{S}}$. So the other sets in the cover need only cover a finite number of points in A. Hence the open cover has a finite subcover, and A is compact. □

The set Φ is the set of (Borel) probability measures on A. Since A is metric and compact, Φ is metrizable and weakly compact (see Hildenbrand, 1974, p. 49). Note that Φ is a subset of ℓ^1, a Hausdorff topological linear space. It is straightforward to show that Φ is convex. Of course Φ^{I+J} shares all these properties.

Claim 2. *For any $k\in K$, the mapping $\pi^k: A\times\Phi^{I+J}\to\mathbb{R}$ is continuous over A.*

Proof. Consider a buyer of valuation u_i. Fix any $\phi\in\Phi^{I+J}$. Let C be any open subset of $\mathbb{R}$. If $\pi^i(\infty;\phi)\notin C$, then the inverse image of C (projected onto A) is a subset of $\mathbb{N}_{\mathrm{VS}}\cup\mathbb{N}_{\mathrm{S}}$, hence an element of $\mathcal{T}$, hence open. Now suppose $\pi^i(\infty;\phi)\in C$. A glance at the definition of π^i shows that

$$\lim_{a_{\mathrm{VS}}\to\infty}\pi^i(a_{\mathrm{VS}})=\lim_{a_{\mathrm{S}}\to\infty}\pi^i(a_{\mathrm{S}})=\pi^i(\infty).$$

It follows that for t_{VS} and t'_{S} large enough, $\pi^i(a_{\mathrm{VS}};\phi)\in C$ and $\pi^i(a'_{\mathrm{S}};\phi)\in C$ for all $a\geq t$ and for all $'a\geq t'$. Consequently the inverse image of C (projected onto A) contains the set of points $\{t_{\mathrm{VS}},(t+1)_{\mathrm{VS}},\ldots\}$, and the set of points $\{t'_{\mathrm{S}},(t'+1)_{\mathrm{S}},\ldots\}$ as well as ∞, and therefore is an element of $\mathcal{T}$ and an open set. The same holds for the other types of agents. This establishes continuity over A. □

For all $k\in K$, define the best-response correspondence $\psi^k:\Phi^{I+J}\Rightarrow A$ as $\psi^k(\phi)=\arg\max_{b\in A}\pi^k(b;\phi)$. Since A is compact and π^k is continuous over A, we have by Weierstrass' theorem that for any $\phi \in \Phi^{I+J}$, $\psi^k(\phi)$ is nonempty. For all $k \in K$, define $\theta^k : \Phi^{I+J} \Rightarrow \Phi$ as $\theta^k(\phi) = \{\mu \in \Phi \mid \mu(\psi^k(\phi)) = 1\}$. To be an element of $\theta^k(\phi)$, a probability measure μ must put positive mass only on best-response strategies: $\theta^k(\phi)$ is in a sense a set of optimal mixed strategies (which also includes pure strategies). We must keep in mind, however, that these "probabilities" are really fractions of the initial type-k population. Since $\psi^k(\phi)$ is nonempty for any $\phi \in \Phi^{I+J}$, so is $\theta^k(\phi)$. It is straightforward to show that $\theta^k(\phi)$ is also convex-valued.

Claim 3. *For all $k \in K$, the correspondence $\theta^k(\phi)$ is upper-hemicontinuous.*

Proof. Consider a sequence $\{\phi_n\}_{n=1}^{\infty}$ such that $\phi_n \in \Phi^{I+J}$ for all n and $\phi_n \xrightarrow{w} \bar{\phi}$ (where $\xrightarrow{w}$ signifies weak convergence). Consider another sequence $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu_n \in \theta^k(\phi_n)$ for all n and $\mu_n \xrightarrow{w} \bar{\mu}$. Since Φ is weakly compact, we know $\bar{\mu} \in \Phi$. We need to show $\bar{\mu} \in \theta^k(\bar{\phi})$, i.e. $\bar{\mu}(\psi^k(\bar{\phi})) = 1$.

Since A is normal and $\mu_n(\psi^k(\phi_n)) = 1$ for all n, $\bar{\mu}(\limsup_n \psi^k(\phi_n)) = 1$ (see Khan, 1989, Lemma 2).

If we fix $a \in A$ and let ϕ vary, $\pi^k(a; \phi)$ becomes a bounded linear functional on Φ^{I+J}, hence an element of the dual space $(\Phi^{I+J})'$. Since $\phi_n \xrightarrow{w} \bar{\phi}$, we have $\pi^k(a; \phi_n) \to \pi^k(a; \bar{\phi})$ for all $a \in A$.

Suppose $a \in A \backslash \psi^k(\bar{\phi})$. Then $\pi^k(a; \bar{\phi}) < \pi^k(b; \bar{\phi})$ for some $b \in A$. There must be m such that $\pi^k(a; \phi_n) < \pi^k(b; \phi_n)$ for all $n > m$. This means that $a \in \liminf_n (A \backslash \psi^k(\phi_n))$, and thus $(A \backslash \psi^k(\bar{\phi})) \subset \liminf_n (A \backslash \psi^k(\phi_n))$. And since $\liminf_n (A \backslash \psi^k(\phi_n)) \cap \limsup_n \psi^k(\phi_n) = \emptyset$, it follows that $\limsup_n \psi^k(\phi_n) \subset \psi^k(\bar{\phi})$.

By monotonicity of probability measures, $\bar{\mu}(\limsup_n \psi^k(\phi_n)) \leq \bar{\mu}(\psi^k(\bar{\phi}))$. Consequently $\bar{\mu}(\psi^k(\bar{\phi})) = 1$. □

Finally, define $\theta : \Phi^{I+J} \Rightarrow \Phi^{I+J}$ as $\theta(\phi) = \times_{k \in K} \theta^k(\phi)$. We know that θ^k is upper-hemicontinuous, convex-valued and nonempty-valued, for all $k \in K$. These properties extend to θ. Also, Φ^{I+J} is a nonempty, weakly compact subset of a Hausdorff topological linear space. Hence θ meets the conditions of the Fan–Glicksberg fixed point theorem (Fan, 1952; Glicksberg, 1952), and there exists $\phi \in \Phi^{I+J}$ such that $\phi \in \theta(\phi)$. Such a ϕ satisfies (2) and is therefore a Nash equilibrium of the game Γ.

5. A partial characterization of equilibrium behavior

Having already established existence, in this section we present a partial characterization of equilibrium behavior for a fixed value of the discount factor δ. It turns out that traders can be classified in four groups within each side.

(i) Intramarginal traders: buyers with valuation $u_i < p^{\mathrm{W}}$ and sellers with cost $c_j > p^{\mathrm{W}}$. They would not trade at all in the competitive mechanism.

(ii) Marginal traders: buyers and sellers with values exactly equal to p^{W}. Since they are indifferent between not trading and trading at p^{W}, some of them should not trade in the competitive mechanism.

(iii) Low surplus traders: buyers with valuation $u_i \in (p^{\mathrm{W}}, p^{\mathrm{H}})$ and sellers with cost $c_j \in (p^{\mathrm{L}}, p^{\mathrm{W}})$. In the competitive mechanism, they would trade at the equilibrium price, but at a "low" surplus. For them, trade at some prices in the game Γ may be unprofitable.

(iv) High surplus traders: buyers with valuation $u_i \geq p^{\mathrm{H}}$ and sellers with cost parameter $c_j \leq p^{\mathrm{L}}$. These are the traders behind the first units in the demand and supply curves, respectively. They would never lose by trading at any price in the market game Γ and they would not gain by not trading.

Our next result shows that for fixed I, J and δ, the equilibrium behavior of these types of traders can be neatly identified.

Proposition 2. *Fix I, J and δ. Every Nash equilibrium ϕ of the game Γ is outcome equivalent to one in which*:

(i) *For intramarginal traders of type k,*

$$\arg\max_a \pi^k(a; \phi) = \{\infty\}.$$

(ii) *For marginal traders of type k with valuation or cost equal to p^W,*

$$\arg\max_a \pi^k(a; \phi) \subseteq \mathbb{N}_S \cup \{\infty\}.$$

(iii) *For low surplus traders of type k,*

$$\arg\max_a \pi^k(a; \phi) \subseteq \mathbb{N}_S.$$

(iv) *For high surplus traders of type k,*

$$\arg\max_a \pi^k(a; \phi) \subseteq \mathbb{N}_{VS} \cup \mathbb{N}_S.$$

Proof. The proof is organized in several lemmata. □

Lemma 1.

(a) *If buyers with valuation $u_i < p^W$ trade, they do so at p^L with probability* 1.

(b) *If sellers with cost $c_j > p^W$ trade, they do so at p^H with probability* 1.

Proof. We prove (a) and omit the identical proof of (b). Suppose not. That is, suppose that for these buyers equilibrium trade takes place after they switch with ex ante positive probability. Then, we shall show that infinity is for them a better response than switching. We establish this in two simple steps.

(i) $\pi^i(n_{\mathrm{VS}}; \phi) < \pi^i(n_{\mathrm{S}}; \phi)$. To see this, note that the payoff to both strategies during the "tough" phase is identical. On the other hand, conditional on reaching period n, a switch to "very soft" in period n entails trading at p^{LH} with probability σ_n^{VS}, at p^{WH} with probability σ_n^{S} and at p^{H} with probability $1-\sigma_n^{\mathrm{VS}}-\sigma_n^{\mathrm{S}}$. If instead, conditional on reaching period n, a buyer switched to "soft" in period n, he would trade at p^{LW} with probability σ_n^{VS}, at p^{W} with probability σ_n^{S}, while he would disagree in period n with probability $1-\sigma_n^{\mathrm{VS}}-\sigma_n^{\mathrm{S}}$. But after disagreeing, he would keep trading after period n upon meeting a "soft" or "very soft" seller, and these trades would take place at prices as high as p^{W} instead of at p^{H} (the price at which he would have traded in period n upon meeting a "tough" seller). Therefore, the expected payoff to playing n_{S} exceeds the one from playing n_{VS}.

(ii) $\pi^i(n_{\mathrm{S}};\phi) < \pi^i(\infty;\phi)$. To see this, note that by switching to soft at some point, this buyerwould trade only either at p^{W} when meeting a soft seller or at p^{LW} when meeting a very soft seller. By never switching and maintaining the tough position, he would avoid the loss when meeting a soft seller and trade at p^{L} upon meeting a very soft one. □

Therefore, we have shown that any Nash equilibrium of the game is outcome equivalent to one where every intramarginal trader plays ∞.

Lemma 2.

(a) *Consider a buyer with valuation* $u_i \in [p^W, p^H)$. *Then,*

$$n_{VS} \notin \arg\max_a \pi^i(a;\phi)$$

for any $n_{VS} \in \mathbb{N}_{VS}$.

(b) *Consider a seller with cost* $c_j \in (p^L, p^W]$. *Then,*

$$n_{VS} \notin \arg\max_a \pi^j(a;\phi)$$

for any $n_{VS} \in \mathbb{N}_{VS}$.

Proof. Recall that we have assumed that the set of intramarginal traders is nonempty. Then, given that they play infinity and that they are present in both sides of the market, the probability that the traders in the statement of the present lemma reach any period n is positive if they play strategy n_{VS}. Then, the rest of the proof is identical to argument (i) in the proof of Lemma 1. □

Lemma 3.

(a) *Consider a buyer with valuation* $u_i \in (p^W, p^H]$. *Then,*

$$\infty \notin \arg\max_a \pi^i(a;\phi).$$

(b) *Consider a seller with cost* $c_j \in [p^L, p^W)$. *Then,*

$$\infty \notin \arg\max_a \pi^j(a;\phi).$$

Proof. We prove part (a) and omit the identical proof of (b). The proof follows from the fact that $\pi^i(\cdot;\phi)$ is continuous at ∞.

By contradiction, suppose infinity is one of their best responses. We first show that $\{\infty\} \neq \arg\max_a \pi^i(a;\phi)$. Note that

$$\begin{aligned}&\lim_{n\to\infty} \pi^i(n_{\mathrm{S}};\phi) - \pi^i(\infty;\phi)\\ &\quad = \lim_{n\to\infty} \Pi_{r=0}^{n-1}(1-\sigma_r^{\mathrm{VS}}) \times \left[\sum_{t=n}^{\infty} \Pi_{r=n}^{t-1}(1-\sigma_r^{\mathrm{VS}}-\sigma_r^{\mathrm{S}})\delta^t(\sigma_t^{\mathrm{VS}}(u_i - p^{\mathrm{LW}})\right.\\ &\qquad \left. + \sigma_t^{\mathrm{S}}(u_i - p^{\mathrm{W}}))\right] = 0+.\end{aligned}$$

(To get this expression, just recall the equations for π^i). This implies that some n_{S} big enough must also be a best response for these buyers.

The argument in the previous paragraph also implies that for every $n_{\mathrm{S}} \in \mathbb{N}_{\mathrm{S}}$, there exists $n' > n$ such that $n'_{\mathrm{S}} \in \arg\max_a \pi^i(a;\phi)$. Without loss of generality, suppose that $\{n_{\mathrm{S}}, (n+1)_{\mathrm{S}}, \ldots, \infty\} \subseteq \arg\max_a \pi^i(a;\phi)$ for some $n_{\mathrm{S}} \in \mathbb{N}_{\mathrm{S}}$. It follows that for every $t > n$, there exists $t' \geq t$ such that $\sigma_{t'}^{\mathrm{VS}} + \sigma_{t'}^{\mathrm{S}} > 0$: to make t_{S} a best response and compensate for discounting, matchings where trade takes place after switching are required. Since the very soft sellers exit as soon as they switch (unlike the soft sellers), $\sigma_t^{\mathrm{VS}} \to 0$ as $t \to \infty$ much faster than σ_t^{S}. It follows that for every $t > n$, there exists $t' > t$ such that $\sigma_{t'}^{\mathrm{S}} > 0$. Also, because ∞ is a best response for these buyers, it follows that for every $t > n$, there exists $t' > t$ such that $\Pi_{r=0}^{t'}(1-\sigma_r^{\mathrm{VS}}-\sigma_r^{\mathrm{S}}) > 0$. But then, recalling that these buyers' valuation strictly exceeds p^{W}, we have that the sign of $\pi^i(t'_{\mathrm{S}};\phi) - \pi^i(\infty;\phi)$ is equal to that of:

$$\Pi_{r=0}^{t'-1}(1-\sigma_r^{\mathrm{VS}}) \left[\sum_{h=t'}^{\infty} \Pi_{r=t'}^{h-1}(1-\sigma_r^{\mathrm{VS}}-\sigma_r^{\mathrm{S}})\delta^{\mathrm{H}}(\sigma_h^{\mathrm{VS}}(u_i - p^{\mathrm{LW}}) + \sigma_h^{\mathrm{S}}(u_i - p^{\mathrm{W}}))\right] > 0$$

for large enough t', which is a contradiction. □

Lemma 4.

(a) *Consider a buyer with valuation $u_i > p^H$. Then,*

$$\infty \notin \arg\max_a \pi^i(a;\phi).$$

(b) *Consider a seller with cost $c_j < p^L$. Then,*

$$\infty \notin \arg\max_a \pi^j(a;\phi).$$

Proof. We prove part (a) and omit the identical proof of (b). We show that $\pi^i(\infty,\phi) < \pi^i(a_{\mathrm{VS}};\phi)$ for some large enough a. Note that $\pi^i(\infty;\phi)$ results from a trading probability which is bounded above by Q^{W}, since it is guaranteed that, upon meeting an intramarginal seller, no trade will result. On the other hand, by playing a_{VS} for any a, the trading probability is 1. It therefore suffices to choose a large enough so that the payoff from the "tough" phase is close enough to the payoff from playing ∞. When this difference has been made very small, this buyer can switch to "very soft" and trade at a positive surplus. We first show a preliminary step.

Claim. *There exists t large enough such that* $\phi^S(t'_{VS} \geq t_{VS}) = 0$ *and* $\phi^B(t'_V \geq t_{VS}) = 0$.

To prove the claim, first recall that, by the previous lemmas, the only traders playing a "very soft" strategy are at most those sellers with cost no greater than p^{L} and those buyers with valuation no lower than p^{H}. We argue by contradiction: suppose not, and without loss of generality, assume that for all t, $\phi^{\mathrm{S}}(t_{\mathrm{VS}}) > 0$ and $\phi^{\mathrm{B}}(t_{\mathrm{VS}}) > 0$.[11] This implies that $\sigma_t^{\mathrm{VS}} > 0$ and $\rho_t^{\mathrm{VS}} > 0$ for all t. Also, "very soft" traders exit the market as soon as they switch, while this is not the case for the rest of traders. Hence, market demographics implies that traders playing a "very soft" strategy exit at a much faster rate than that of reduction of market size. It follows that for every $\epsilon > 0$, there exists t large enough so that $\sigma_t^{\mathrm{VS}} < \epsilon$ and $\rho_t^{\mathrm{VS}} < \epsilon$.

Consider now a typical seller with cost $c_j \leq p^{\mathrm{L}}$ (these are the only sellers who could be playing very soft) and write the difference of payoffs:

$$\begin{aligned}
&\pi^j((t+1)_{\mathrm{VS}}) - \pi^j(t_{\mathrm{VS}}) \\
&\quad = \delta^t \Pi_{r=0}^{t-1}(1-\rho_r^{\mathrm{VS}})[\rho_t^{\mathrm{VS}}(p^{\mathrm{H}}-c_j) + \delta(1-\rho_t^{\mathrm{VS}})(\rho_{t+1}^{\mathrm{VS}}(p^{\mathrm{LH}}-c_j) \\
&\qquad +(1-\rho_{t+1}^{\mathrm{VS}})(p^{\mathrm{L}}-c_j)) - (\rho_t^{\mathrm{VS}}(p^{\mathrm{LH}}-c_j) + (1-\rho_t^{\mathrm{VS}})(p^{\mathrm{L}}-c_j))] < 0
\end{aligned}$$

because $\delta < 1$ is fixed. Therefore, no seller from any of these types would like to play $(t+1)_{\mathrm{VS}}$ contradicting that $\phi^{\mathrm{S}}((t+1)_{\mathrm{VS}}) > 0$. This establishes the claim.

To finish the proof of the lemma, write down the payoff difference between playing a_{VS} and playing ∞ for a buyer with valuation $u_i > p^{\mathrm{H}}$:

$$\begin{aligned}
\pi^i(a_{\mathrm{VS}}) - \pi^i(\infty) = -\sum_{t=a}^{\infty} \delta^t \Pi_{r=0}^{t-1}(1-\sigma_r^{\mathrm{VS}})\sigma_t^{\mathrm{VS}}(u_i - p^{\mathrm{L}}) + \delta^a \Pi_{r=0}^{a-1}(1-\sigma_r^{\mathrm{VS}}) \\
\times [\sigma_a^{\mathrm{VS}}(u_i - p^{\mathrm{LH}}) + \sigma_a^{\mathrm{S}}(u_i - p^{\mathrm{WH}}) + (1-\sigma_a^{\mathrm{VS}} - \sigma_a^{\mathrm{S}})(u_i - p^{\mathrm{H}})],
\end{aligned}$$

which, using the earlier claim, is greater than 0 for some large enough a.

This ends the proof of Proposition 2.

Notice the "monotonicity" in the behavior of traders in the game. Take the buyers' side, for example. The intramarginal buyers would always play tough and therefore trade only at the lowest possible price p^{L}. The low surplus buyers play soft at some point: hence, they trade at p^{L} during their tough phase and at prices as high as p^{W} after they switch. Marginal buyers, instead, either play soft or may find it worthwhile never to switch to a soft or very soft position. Finally, the high surplus buyers have more flexibility and always switch to either a soft or a very soft position: they may end up trading at any price allowed in the game.

6. Examples of equilibria

The previous section has offered a useful characterization of equilibrium behavior for a fixed value of the discount factor δ. To explore the possibilities offered by the model, this

[11] The other cases in which there exists t such that $\phi^{\mathrm{S}}(t'_{\mathrm{VS}} \geq t_{\mathrm{VS}}) = 0$, while $\phi^{\mathrm{B}}(t'_{\mathrm{VS}}) > 0$ for all t' or vice versa are clearly impossible, since these traders have no incentive to keep playing tough once the other side does not play very soft at all.

section is devoted to the study of some examples, describing equilibria that exist when δ is arbitrarily close to 1. The first yields the Walrasian outcome, while the others do not.

Example 1. Consider the profile of strategies ϕ, where

(i) $\phi^i(0_S) = 1$ for every i such that $u_i > p^W$;
(ii) $\phi^i(\infty) = 1$ for every i such that $u_i < p^W$;
(iii) $\phi^j(0_S) = 1$ for every j such that $p^W > c_j$;
(iv) $\phi^j(\infty) = 1$ for every j such that $p^W < c_j$;
(v) $\phi^i(0_S) = \mu$ and $\phi^i(\infty) = 1 - \mu$ for those i with $u_i = p^W$, where μ plus the mass of buyers in (i) equates the mass of sellers in (iii).

We shall argue that there exists δ close enough to 1 such that the profile ϕ is a Nash equilibrium of the game Γ for every $\delta' > \delta$ and for every I and J.

By Proposition 2, those buyers in part (ii) of the strategies in the statement and those sellers in part (iv) are at a best response. They never trade and their equilibrium payoff is 0.

Buyers in part (v) obtain an equilibrium payoff exactly equal to 0 (they are indifferent between not trading and trading at p^W, the only price at which transactions take place in equilibrium). It is easy to see that no unilateral deviation pays for these agents, either.

Finally, for δ close enough to 1, buyers in (i) and sellers in (iii) obtain an equilibrium expected payoff arbitrarily close to $u_i - p^W$ and $p^W - c_j$, respectively. To show this, it suffices to show that their trading probability, given the proposed equilibrium strategies, is 1. Since this probability is independent of the discount factor (because the strategies do not vary with it), as δ goes to 1, we have that for all $\epsilon > 0$, we can find a period t such that the sum of expected payoffs over the first t periods is within an ϵ-ball around $u_i - p^W$ for the buyers and within an ϵ-ball around $p^W - c_j$ for the sellers.

Therefore, we need to prove that, given the proposed strategies, the trading probability for a buyer in (i) is 1 (the argument for sellers in (iii) is identical and will be omitted). Note that, every period exactly a mass $1 - Q^W$ of sellers play "tough." However, this becomes an increasing proportion of the market, whose size is N_t in period t. Therefore, the probability of "no trade" for a buyer in (i) is:

$$p_{NT} = \Pi_{t=0}^{\infty} \frac{1 - Q^W}{N_t}.$$

We need to show that $p_{NT} = 0$. To see this, observe that, given the proposed strategies, the law of motion of the size of the market is:

$$N_{t+1} = N_t - \frac{[N_t - (1 - Q^W)]^2}{N_t}.$$

That is, the buyers who remain in the market from one period to the next are those who do not trade. The ones who trade are found among the $N_t - (1 - Q^W)$. Each of them trades if and only if he is matched to a soft seller. This event in period t occurs with probability $[N_t - (1 - Q^W)]/N_t$. Hence the above expression.

Operating, we can express the law of motion of N_t as follows:

$$N_{t+1} = 2(1 - Q^{\mathrm{W}}) - \frac{(1 - Q^{\mathrm{W}})^2}{N_t}.$$

We next proceed to a change in variables: let $M_t = N_t/(1 - Q^{\mathrm{W}})$. Therefore:

$$M_{t+1} = 2 - \frac{1}{M_t} = \frac{2M_t - 1}{M_t}.$$

We also have that:

$$M_{t+2} = 2 - \frac{1}{M_{t+1}} = \frac{3M_t - 2}{2M_t - 1},$$

and so on, so that:

$$M_{t+n} = \frac{(n+1)M_t - n}{nM_t - (n-1)}.$$

Note that the probability of "no trade" is

$$p_{\mathrm{NT}} = \Pi_{t=0}^{\infty} \frac{1}{M_t} = \lim_{n\to\infty} \frac{1}{M_0} \frac{M_0}{2M_0 - 1} \frac{2M_0 - 1}{3M_0 - 2} \cdots \frac{nM_0 - (n-1)}{(n+1)M_0 - n}.$$

Given the multiple cancellations, we have that:

$$p_{\mathrm{NT}} = \lim_{n\to\infty} \frac{1}{(n+1)M_0 - n} = \lim_{n\to\infty} \frac{1/n}{M_0 - 1} = 0.$$

Therefore, the probability of trade, given the proposed strategies for a buyer in part (i) (and for a seller in part (iii)) is 1. Thus, their equilibrium payoffs, for δ large enough, are "close" to $u_i - p^{\mathrm{W}}$ and $p^{\mathrm{W}} - c_j$, respectively. It follows that no unilateral deviation is profitable for these agents. Switching to "very soft" entails immediate trade, but at less favorable prices. Switching to "tough" only produces delay since no agent on the other side plays "very soft."

Therefore, we have constructed an equilibrium for δ large enough that yields (approximately) the Walrasian outcome. All transactions take place at p^{W}, a measure Q^{W} of units are traded and (asymptotic as $\delta \to 1$) efficiency obtains: all traders with positive gains from trade at the competitive price end up trading and getting their competitive surplus.

We turn now to the other possibilities offered by the model. The other equilibria we present in this section, which also remain as $\delta \to 1$, yield non-Walrasian outcomes. But these are of very different nature. While in Example 2 the main theme is rationing due to the existence of a finite number of types of traders, Example 3 concerns the inefficient equilibria that can be created by the large degree of flexibility in the hands of high surplus traders. Finally, a coordination failure underlies Example 4, based on delay.

Example 2. Denote by i^* the type of buyer for whom $u_{i^*} = p^{\mathrm{W}}$. Suppose that for all $i < i^*$,

$$\frac{Q^{\mathrm{W}}}{\sum_{i=1}^{i*}\alpha_i}(u_i - p^{\mathrm{W}}) > Q^{\mathrm{W}}(u_i - p^{\mathrm{WH}}) + (1 - Q^{\mathrm{W}})(u_i - p^{\mathrm{H}}).$$

Consider the profile of strategies ϕ, where

(i) $\phi^i(0_{\mathrm{S}}) = 1$ for every i such that $u_i \geq p^{\mathrm{W}}$;
(ii) $\phi^i(\infty) = 1$ for every i such that $u_i < p^{\mathrm{W}}$;
(iii) $\phi^j(0_{\mathrm{S}}) = 1$ for every j such that $p^{\mathrm{W}} > c_j$;
(iv) $\phi^j(\infty) = 1$ for every j such that $p^{\mathrm{W}} < c_j$.

We shall argue that there exists δ close enough to 1 such that the profile ϕ is a Nash equilibrium of the game Γ for every $\delta' > \delta$.

The arguments to show that the buyers in (ii) and the sellers in (iv) are best-responding when using their proposed strategy are identical to those outlined in Proposition 2. For sellers in (iii), as their trading probability continues to be 1, the argument is provided in Example 1. The only change now is that the trading probability for buyers in (i) is no longer 1. In order to exactly quantify it, we go through similar steps as in Example 1.

Given the proposed strategies, the law of motion of the size of the market is:

$$N_{t+1} = N_t - \frac{[N_t - (1 - Q^{\mathrm{W}})][N_t - (1 - \sum_{i=1}^{i*}\alpha_i)]}{N_t}.$$

That is, the buyers who remain in the market from one period to the next are those who do not trade. The ones who trade are found among the $N_t - (1 - \sum_{i=1}^{i*}\alpha_i)$ who play "soft" in period t. Each of them trades if and only if he is matched to a soft seller. This event in period t occurs with probability $[N_t - (1 - Q^{\mathrm{W}})]/N_t$. Hence the above expression.

Operating, we can express the law of motion of N_t as follows:

$$N_{t+1} = (1 - Q^{\mathrm{W}}) + \left(1 - \sum_{i=1}^{i*}\alpha_i\right) - \frac{(1 - Q^{\mathrm{W}})(1 - \sum_{i=1}^{i*}\alpha_i)}{N_t}.$$

Proceeding to the same change in variables as in Example 1, let $M_t = N_t/(1 - Q^{\mathrm{W}})$. Also, let $\omega = (1 - \sum_{i=1}^{i*}\alpha_i)/(1 - Q^{\mathrm{W}})$. Therefore:

$$M_{t+1} = 1 + \omega - \frac{\omega}{M_t} = \frac{(1+\omega)M_t - \omega}{M_t}.$$

We also have that:

$$M_{t+2} = 1 + \omega - \frac{\omega}{M_{t+1}} = \frac{(1+\omega+\omega^2)M_t - (\omega+\omega^2)}{(1+\omega)M_t - \omega},$$

and so on, so that:

$$M_{t+n} = \frac{(1+\omega+\omega^2+\omega^n)M_t - (\omega+\omega^2+\omega^n)}{(1+\omega+\omega^2+\omega^{n-1})M_t - (\omega+\omega^2+\omega^{n-1})}.$$

Recall that the probability of "no trade" is

$$p_{\mathrm{NT}} = \Pi_{t=0}^{\infty} \frac{1}{M_t} = \lim_{n\to\infty} \frac{1}{M_0} \frac{M_0}{(1+\omega)M_0-\omega} \frac{(1+\omega)M_0-\omega}{(1+\omega+\omega^2)M_0-(\omega+\omega^2)} \cdots \times \frac{(1+\omega+\omega^2+\cdots+\omega^{n-1})M_0-(\omega+\cdots+\omega^{n-1})}{(1+\cdots+\omega^n)M_0-(\omega+\cdots+\omega^n)}.$$

Given the multiple cancellations, we have that:

$$p_{\mathrm{NT}} = \lim_{n\to\infty} \frac{1}{(1+\cdots+\omega^n)M_0-(\omega+\cdots+\omega^n)} = \lim_{n\to\infty} \frac{1-\omega}{M_0-\omega} = \frac{\sum_{i=1}^{i^*}\alpha_i - Q^{\mathrm{W}}}{\sum_{i=1}^{i^*}\alpha_i}.$$

Hence, the trading probability for buyers in (i) is $Q^{\mathrm{W}}/\sum_{i=1}^{i^*}\alpha_i$. It follows that their expected payoff to playing strategy 0_{S} approaches $\frac{Q^{\mathrm{W}}}{\sum_{i=1}^{i^*}\alpha_i}(u_i - p^{\mathrm{W}})$ as $\delta \to 1$. It can be checked that the best alternative to this strategy is to play 0_{VS} instead, whose expected payoff is $Q^{\mathrm{W}}(u_i - p^{\mathrm{WH}}) + (1-Q^{\mathrm{W}})(u_i - p^{\mathrm{H}})$. Given the inequality assumed at the beginning of the example, these buyers are also at a best response for some large enough δ.

The equilibrium identified in Example 2 provides a subtle reason for a non-Walrasian outcome. Note that in the equilibrium: (a) the output exchanged V is the Walrasian Q^{W}; and (b) all trade takes place at the Walrasian price p^{W}, i.e. $V_{\mathrm{W}} = 1$. However, the outcome is not Walrasian as there is a loss of asymptotic efficiency. Namely, some buyers with valuation $u_i > p^{\mathrm{W}}$ do not trade as they are crowded out by other buyers with valuation $u_i = p^{\mathrm{W}}$. The latter are indifferent between not trading and doing so at p^{W}. When choosing to trade in the above equilibrium, rationing must take place since there are only Q^{W} units for sale. Given the private values problem (the planner does not identify valuations), the mentioned inefficiency arises. note finally that this inefficiency is an artefact of there being “many” traders with valuation exactly equal to the competitive price.

Example 3. For simplicity, suppose $I = J = 2$ and consider the following values of the parameters, which fall under our general assumptions: $u_1 > u_2 = p^{\mathrm{W}}$ and $c_1 < p^{\mathrm{W}} < c_2$. In addition, we shall assume to simplify computations that the model is symmetric in some respects, i.e. let $\alpha_1 = \beta_1 = q$, and assume that payoffs satisfy that $u_1 - p^{\mathrm{L}} = p^{\mathrm{H}} - c_1 = D_1$, $u_1 - p^{\mathrm{LH}} = p^{\mathrm{LH}} - c_1 = D_2$ and $u_1 - p^{\mathrm{H}} = p^{\mathrm{L}} - c_1 = D_3$. With this specification, the competitive output is a measure $Q^{\mathrm{W}} = q$. Any price in the interval $[u_2, c_2]$ is Walrasian. For the sake of argument, suppose that at least the extreme prices utilized in the model, p^{H} and p^{L}, do not fall on this interval. In the competitive equilibrium, all buyers with valuation u_1 should trade with all sellers of cost c_1 at one of the competitive prices, while the other agents do not trade.

We make the dependence of the game on the discount factor explicit because the strategies will be non-constant functions of δ. Consider the following strategies played in the game $\Gamma(\delta)$:

(i) $\phi^i(a_{\mathrm{VS}}) = \phi(a) > 0$, $0 \le a \le T(\delta)$ and $\phi^i([0_{\mathrm{VS}}, T(\delta)_{\mathrm{VS}}]) = 1$ for every buyer of valuation $u_i = u_1$;
(ii) $\phi^i(\infty) = 1$ for every buyer of valuation $u_i = u_2$;
(iii) $\phi^j(a_{\mathrm{VS}}) = \phi(a) > 0$, $0 \le a \le T(\delta)$ and $\phi^j([0_{\mathrm{VS}}, T(\delta)_{\mathrm{VS}}]) = 1$ for every seller of cost $c_j = c_1$;

(iv) $\phi^j(\infty) = 1$ for every seller with cost $c_j = c_2$.

According to these strategies, symmetric for the two sides of the market, a measure $1-(1-q)^2$ of units are traded. A measure q^2 of them are traded at p^{LH}, a measure $q(1-q)$ at p^{H} and a measure $q(1-q)$ at p^{L}. Clearly, this outcome is far from being Walrasian.

Recall that $\phi^i(a)$ is the fraction of initial buyers who play strategy a; similarly for $\phi^j(a)$ and sellers. Note that $\phi^i(\infty) = 1-q$ and $\phi^j(\infty) = 1-q$ according to these strategies.

To see that this strategy profile is an equilibrium, consider the following construction. We must create market conditions so that $0_{\mathrm{VS}}, \dots, T(\delta)_{\mathrm{VS}}$ are all best responses for buyers of valuation u_1 and sellers of cost c_1. Given these strategies, $\sigma_0^{\mathrm{VS}} = \rho_0^{\mathrm{VS}} = \phi(0)$. Moreover, we get that $(1-\sigma_0^{\mathrm{VS}})\sigma_1^{\mathrm{VS}} = (1-\rho_0^{\mathrm{VS}})\rho_1^{\mathrm{VS}} = \phi(1)$, and so on. This fact will simplify the next equations, which can be shown to be implied by the equality of expected payoff to these traders from playing 0_{VS} and 1_{VS}, and so on, until the payoff equality between playing $(T(\delta)-1)_{\mathrm{VS}}$ and $T(\delta)_{\mathrm{VS}}$. We suppress the subscript VS in the equations , since no traders play soft according to the proposed strategy profile.

$$\phi(0)D_2 + [\phi(t \geq 1) + \phi(\infty)]D_3 = \phi(0)D_1 + \delta\phi(1)D_2 + \delta[\phi(t \geq 2) + \phi(\infty)]D_3$$
$$\dots$$
$$\phi(T(\delta)-1)D_2 + [\phi(T(\delta)) + \phi(\infty)]D_3 = \phi(T(\delta)-1)D_1 + \delta\phi(T(\delta))D_2 + \delta(1-q)D_3$$
$$\phi([0, T(\delta)]) = q.$$

Let δ be big enough so that $D_3 < \delta D_2$. It will be convenient to write the typical equation $(t = 1, \dots, T(\delta))$ in the above system as:

$$\phi(t-1)(D_2 - D_1) + \phi(t)(D_3 - \delta D_2) + (1-\delta)[\phi(t' \geq t+1) + \phi(\infty)]D_3 = 0. \quad (*)$$

Also, to be an equilibrium, these traders must prefer weakly to play $T(\delta)_{\mathrm{VS}}$ to their best deviation, which turns out is $(T(\delta)+1)_{\mathrm{VS}}$—deviating to a “soft” strategy or to infinity yields trade with probability less than 1. These considerations yield the following inequality:

$$\phi(T(\delta))(D_2 - D_1) + (1-\delta)(1-q)D_3 \geq 0. \quad (**)$$

Therefore, choose $\phi(T(\delta))$ small enough so that (**) holds, and for each such value of $\phi(T(\delta))$, find from (*) the unique value of $\phi(T(\delta)-1)$, and so on, all the way back to period 0. Note that, for δ large enough, the values of $\phi(t)$ are quite small. To see this, notice for example that if $\phi(T(\delta)) = 0$, we get that $\phi(T(\delta)-1) = ((1-\delta)(1-q)D_3)/(D_1 - D_2)$. If δ is close to 1, all these fractions are small. In addition, all the functions in these equations are continuous so that a solution to the system is found.

Note that, as $\delta \to 1$, $\phi(t) \to 0$ for all $t = 0, \dots, T(\delta)$, which implies that $T(\delta) \to \infty$.

In Example 3 the non-Walrasian outcome is a consequence of the actions of the high surplus traders. Their surplus is sizeable enough so that it is in their interest to make a gamble. They are willing to eventually switch to a very soft position and avoid the “no trade” outcome (recall that the intramarginal traders always play tough) as long as they are compensated with trade at favorable prices, at least with some probability. It is therefore necessary that a fraction of traders of the opposite side also play very soft. Note finally how the example can be extended to an arbitrary number of types: q would then be the fraction of high surplus traders on each side of the market.

Example 4. Consider the profile of strategies ϕ, where

(i) $\phi^i(t(\delta)_S) = 1$ for every i such that $u_i > p^W$;
(ii) $\phi^i(\infty) = 1$ for every i such that $u_i < p^W$;
(iii) $\phi^j(t(\delta)_S) = 1$ for every j such that $p^W > c_j$;
(iv) $\phi^j(\infty) = 1$ for every j such that $p^W < c_j$;
(v) $\phi^i(t(\delta)_S) = \mu$ and $\phi^i(\infty) = 1 - \mu$ for those i with $u_i = p^W$, where μ plus the mass of buyers in (i) equates the mass of sellers in (iii).

Following arguments identical to those in Example 1, one can argue that the limiting equilibrium payoff of those agents who trade is $\lim_{\delta\to 1}\delta^{t(\delta)}(u_i - p^W)$ for buyers and $\lim_{\delta\to 1}\delta^{t(\delta)}(p^W - c_j)$. Choose $t(\delta)$ for each δ such that for high surplus buyers we have that $\delta^{t(\delta)}(u_i - p^W) \geq u_i - p^H$; while for high surplus sellers we have that $\delta^{t(\delta)}(p^W - c_j) \geq p^L - c_j$. These conditions deter deviations to 0_{VS} on the part of these traders. Since this would be their optimal deviation, the profile ϕ is an equilibrium. But note how different choices of the sequence $t(\delta)$ as $\delta \to 1$ will determine whether or not the corresponding equilibrium sequence is asymptotically efficient.

In Example 4, all transactions take place at p^W, the output exchanged is Q^W and trade occurs only among those agents that trade in the competitive equilibrium. There is a co-ordination failure in the choice of “switching to soft” time. Substantial delay may happen affecting traders’ payoffs.

7. The Asymptotic results

In this final section we identify conditions under which all equilibria of the model yield the Walrasian price. First, we shall concentrate on the model when it becomes frictionless (as $\delta \to 1$) in order to make a fair comparison with the centralized paradigm. But this will not suffice, as Examples 2 and 3 demonstrate. Correspondingly, we introduce two conditions to eliminate equilibria like the ones found in those examples: (1) we shall assume that there are no high surplus traders in at least one side of the market; and (2) we shall let the numbers of types of traders I and J increase to infinity.[12] Also, denote by $\Gamma(\delta, I, J)$ the game with discount factor δ, I types of buyers and J types of sellers, by $E(\delta, I, J)$ the set of its Nash equilibrium payoffs, and by $V(\delta, I, J)$ and $V_W(\delta, I, J)$ the statistics V and V_W corresponding to an equilibrium of $\Gamma(\delta, I, J)$.

Proposition 3. *Suppose there are no high surplus traders in at least one side of the market. Then,*

[12] In fact, it is not required that I and $J \to \infty$. It suffices with the weaker condition stating that the mass of marginal traders becomes negligible.

$$\lim_{\delta\to 1 I,J\to\infty} V(\delta, I, J) = Q^W \quad \textit{and} \quad \lim_{\delta\to 1 I,J\to\infty} V_W(\delta, I, J) = 1.$$

Proof. Suppose that the sellers' side is the one without high surplus traders. It follows from Proposition 2 that for fixed δ, I and J, the fractions $\phi(a_{\mathrm{VS}}) = 0$ for all $a_{\mathrm{VS}} \in \mathbb{N}_{\mathrm{VS}}$. Also by Proposition 2, as $J \to \infty$ the sum of measures of sellers who play a_{S} for some $a_{\mathrm{S}} \in \mathbb{N}_{\mathrm{S}}$ approaches Q^{W}, while a measure approaching $1 - Q^{\mathrm{W}}$ plays ∞. For any buyer with valuation $u_i > p^{\mathrm{W}}$, playing 0_{S} yields over time trade with probability approaching 1, by arguments similar to the ones in Example 1. Moreover, as $\delta \to 1$, it is easy to see that n_{S} (for some $n_{\mathrm{S}} \in \mathbb{N}_{\mathrm{S}}$) are the only best responses for these buyers. This in turn implies that for all sellers with cost $c_j < p^{\mathrm{W}}$, the only best responses are also n'_{S} for some $n'_{\mathrm{S}} \in \mathbb{N}_{\mathrm{S}}$. Therefore, exactly Q^{W} units are traded in equilibrium, and all of them at p^{W}. Furthermore, no intramarginal trader gets to trade. □

8. Concluding remarks

1. There is another sense in which I and $J \to \infty$ is not needed for the Walrasian price–output result. For example, suppose there is only one type of seller with cost $c_j = p^{\mathrm{W}}$. It can be shown that in this model all equilibria yield the Walrasian price as $\delta \to 1$. This is so even for a fixed i. To see how the proof goes, note that, applying Proposition 2, sellers will play either a_{S} or ∞. However, an equilibrium where sellers play infinity is impossible, due to arguments related to the Coase conjecture. That is, if sellers played always tough, they would trade only with high surplus buyers, and they would realize that it pays to switch to soft at some point to be able to trade with the low surplus buyers. Anticipating this fall in prices asked by the sellers, the high surplus buyers would not agree to pay high prices for high values of δ. See Serrano and Yosha (1996b) for the details of this result in a related model.
2. On the other hand, the non-Walrasian result found in Serrano and Yosha (1996b) has been recovered here in a more general model involving an arbitrary finite number of types. The reasons for the non-Walrasian result are three-fold: (a) the rationing imposed on positive surplus traders by the marginal traders of the same side; (b) the fear of high surplus agents of not trading, which makes them settle at less favorable terms, thereby yielding an inefficient outcome where some intramarginal agents get to trade; and (c) coordination failure that may create asymptotic inefficiency through delay. Note finally how if there are no intramarginal traders ($Q^{\mathrm{W}} = 1$) the fear force disappears: by holding a soft position, the probability of trade is 1 because there are no traders playing infinity. Then, there is no need to adopt a very soft position and the Walrasian price obtains once again.
3. The model that we have studied in detail makes some simplifying assumptions. We next discuss how the analysis would extend to a more general model. Consider a large finite set of prices in the interval $[c_1, u_1]$. Let λ be the cardinality of this set. Each trader can announce one of these λ prices in a meeting. The only restriction we impose is that of monotonicity: that is, denoting a buyer's announcements in periods t and $t+1$ by $p^{\mathrm{B}}(t)$ and $p^{\mathrm{B}}(t+1)$, we impose that $p^{\mathrm{B}}(t) \leq p^{\mathrm{B}}(t+1)$ for all t. With similar notation for sellers, we impose that $p^{\mathrm{S}}(t) \geq p^{\mathrm{S}}(t+1)$ for all t. That is, positions are not becoming

tougher over time, and there is no restriction to the number of position switches other than λ. The rules of trade continue to be that exchange takes place at the average of the announced prices if the seller's announcement is no greater than that of the buyer's. There is no trade otherwise. We can model this game as each agent having the set of strategies $A^{\lambda-1}$, where $A = \mathbb{N} \cup \{\infty\}$, i.e. number of periods playing the toughest position followed by number of periods playing the second toughest, and so on, all the way to number of periods taking the second softest position (as soon as a trader adopts the softest position, he exits the market trading at that price). If one of these numbers is infinity, the next numbers become irrelevant. For example, if $\lambda = 3$ as in the model written in the body of the paper, a strategy like (n_1, n_2) means holding the toughest position for $n_1 \geq 0$ initial periods, the soft position for $n_2 \geq 0$ and switching to the very soft position in period $n_1 + n_2$. Payoff functions would be extremely cumbersome to write because they would hold multiple summations. The existence result in Proposition 1 and its proof would still go through essentially without making any changes (only replacing A with $A^{\lambda-1}$). The results of Proposition 2 monotonicity in the toughness would extend, although its statement would be considerably more messy: if one compares two buyers with different valuations, the one with the higher one should always find it optimal to play "at least as tough" as the other one, and similarly for sellers. It is easy to see that the equilibrium supporting the Walrasian outcome still exists, using the same arguments as in Example 1. Similar non-Walrasian equilibria are found, and the effects behind them are the same (a)–(c), as listed in the previous paragraph. And if in any equilibrium all trade takes place at one price, this must be the Walrasian price.

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