Available online at www.sciencedirect.com
ScienceDirect
Mathematical Social Sciences 52 (2006) 288 – 301
mathematical social sciences
www.elsevier.com/locate/econbase

A comparison of the average prekernel and the prekernel

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Received 1 October 2005; received in revised form 1 April 2006; accepted 1 July 2006
Available online 7 September 2006

Abstract

We propose axiomatic foundations for the average prekernel of NTU games, and compare them with the existing ones for the prekernel. We characterize the average prekernel as the unique solution that satisfies a set of Nash-like axioms for two-person games, and versions of average consistency and its converse for multilateral settings.

Keywords: Average prekernel; Prekernel; Nash program; Bargaining; Consistency

JEL classification: C71; C72; C78

1. Introduction

The prekernel of a non-transferable utility (NTU) coalitional game consists of those payoffs in which each player is in a situation of "bilateral equilibrium" with any other player.[1] The prekernel was introduced for the class of transferable utility or TU games in Davis and Maschler (1965), and generalized to the class of NTU games in Moldovanu (1990) and Serrano (1997).[2] The latter paper, as part of the Nash program for coalitional games (Nash, 1950, 1953; see Serrano (2005) for a recent survey), also contained a non-cooperative model of negotiations to support the payoffs in the prekernel. The prekernel was characterized in

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[1]See Maschler (1992) for a survey.
[2]See Kalai (1975) for an early alternative definition.

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doi:10.1016/j.mathsocsci.2006.07.004

Peleg (1986) for the class of TU games and in Serrano and Shimomura (1998) for the class of smooth NTU games.[3]

In multilateral settings, the prekernel is a good description of equilibrium in bilateral bargaining between any pair of players. However, it has an important shortcoming. As pointed out in Moldovanu (1990) and Serrano (1997), it often prescribes the empty set. Following an idea of Maschler and Owen's (1992) for the NTU Shapley value and Dagan and Volij's (1997) for bankruptcy rules, this existence problem has been recently solved in Orshan and Zarzuelo (2000), who propose as an alternative solution concept the *average prekernel*.[4] The average prekernel is the set of efficient payoffs where, for each player, the aggregate (or average) difference of surpluses of a player against all the others is zero. Thus, it always contains the prekernel (because in the latter the difference of surpluses is zero for every pair of players). However, its advantage over the prekernel is that, as Orshan and Zarzuelo (2000) demonstrate, it is non-empty very generally, over a large significant class of NTU games. This result has been refined even further for the intersection of the average prekernel and the core in Orshan et al. (2003), who show this intersection is non-empty when the game is "boundary separating".

It is important to clarify the distinction between prekernel and average prekernel, by proposing strategic and axiomatic analyses of the average prekernel. These ought to be compared with those for the prekernel, contained in Serrano (1997) and Serrano and Shimomura (1998). This paper will focus on the axiomatic foundation, and we comment on its strategic counterpart in our concluding section.

The only differences between the axioms in Serrano and Shimomura (1998) and those used in the present paper are found in the different versions of consistency and its converse. Instead of employing consistency and its converse in the sense of the Davis–Maschler reduced game, we need average consistency and average converse consistency with respect to certain reduced hyperplane games to characterize the average prekernel over the class of smooth NTU games.[5]

The basis of our characterization is the class of two-player games, where we use the same axioms as in Serrano and Shimomura (1998): non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence –we shall motivate each of these axioms after it is formally stated. For more than two players, one needs to make both changes (consistency versus average consistency, and Davis–Maschler reduced games versus reduced hyperplane games) to go from the axioms of the prekernel to those of the average prekernel. Indeed, we show an impossibility result if one works with average consistency using Davis–Maschler reduced games.

Finally, we also offer a parallel result to the last theorem in Serrano and Shimomura (1998).[6] In the last result of the current paper, the intersection of the core and the average prekernel is characterized using the same axioms as for the average prekernel, but for the class of smooth games with non-empty cores. We show that the axioms utilized in all our characterizations are logically independent.

[3] Serrano and Shimomura (1998), borrowing terminology from Maschler et al. (1988), used the name "Nash set" for the prekernel. Although there are some arguments to justify that choice (see Serrano and Shimomura (1998)), we shall stick to the name "prekernel" here.

[4] To continue with our footnotes on terminology, we shall use this name to refer to their solution. The one they used, that of *bilateral consistent prekernel*, is somewhat confusing: first, the average prekernel is not bilaterally consistent; and second, one of the key axioms used in Serrano and Shimomura (1998) to characterize the prekernel is precisely that of bilateral consistency.

[5] See Thomson (1996) for a comprehensive survey on consistency.

[6] That theorem is a characterization of the intersection of the core and the prekernel. Moldovanu (1990) also offers a partial characterization of this intersection for the class of convex NTU assignment problems.

2. Preliminaries

Denote by $\mathbb{R}$ the set of the real numbers. If we use an upper case letter to denote a set, its lower case counterpart denotes its cardinality. Thus, let N be a finite set containing at least two elements, and let $n=|N|$. Denote by $\mathbb{R}^N$ the set of all functions from N to $\mathbb{R}$. We identify an element $x\in\mathbb{R}^N$ with an n-dimensional vector whose components are indexed by members of N; thus we write x_i for $x(i)$. If $x\in\mathbb{R}^N$ and $S\subseteq N$, we write x_S for the restriction of x to S, which is the element of $\mathbb{R}^S$ that associates x_i with each $i\in S$. We also write x_{-S} to denote $x_{N\setminus S}$ and x_{-i} to denote $x_{N\setminus\{i\}}$.

A *player* is an element of N, and a non-empty subset S of N is a *coalition*. A *payoff* to player i is a point of $\mathbb{R}^{\{i\}}$, and a *payoff profile* for coalition S is a point of $\mathbb{R}^S$. The *Pareto frontier* of a set of payoffs $Y\subseteq\mathbb{R}^S$ is:

$$\partial Y = \{y\in Y|\ x_i > y_i \qquad \forall i\in S \qquad \text{implies } x\notin Y\}.$$

Definition 1. The pair (N,V) is a *non-transferable utility coalitional game*, or simply an (NTU) *game*, if V is a correspondence that associates with every non-empty $S\subseteq N$ a non-empty proper subset $V(S)\subset\mathbb{R}^S$ satisfying the following assumptions:

Assumption (1). $V(S)$ is closed. Also, it is comprehensive, i.e., for each $x_S\in V(S)$, $\{x_S\}-\mathbb{R}^S_+\subseteq V(S)$.

Assumption (2). For each $x_S\in\mathbb{R}^S$,

$$\partial V(S)\cap\left(\{x_S\}+\mathbb{R}^S_+\right)$$

is bounded.

Assumption (3). There exists a continuously differentiable representation of $V(N)$, i.e., a continuously differentiable function $g:\mathbb{R}^N\to\mathbb{R}$ such that

$$V(N) = \left\{x\in\mathbb{R}^N|g(x)\leq 0\right\}.$$

Thus, using g, one can write that the interior of $V(N)$ is the set

$$\text{Int}V(N) = \left\{x\in\mathbb{R}^N|g(x) < 0\right\},$$

and the *Pareto frontier* of $V(N)$ is the set $\partial V(N)$ of points $x\in\mathbb{R}^N$ such that $g(x)=0$.

Assumption (4). $V(N)$ is non-levelled, i.e., for every $x\in\partial V(N)$, the gradient of g at x is positive in all its coordinates, i.e., $\nabla g(x)\gg 0$. We shall write $g_i(x)$ for the partial derivative of g at $x\in\partial V(N)$ with respect to component $i\in N$. Thus, $\nabla g(x)=(g_i(x))_{i\in N}$, so $g_i(x)>0$ for all $i\in N$ and for all $x\in\partial V(N)$.

A *transferable utility game*, or a TU game, is a coalitional game (N,V) defined by a function v that associates with every coalition S a real number $v(S)$ such that

$$V(S) = \left\{x_S\in\mathbb{R}^S|\sum_{i\in S} x_i\leq v(S)\right\}.$$

Abusing notation, we use (N,v) to denote the associated coalitional game.

A *hyperplane game* is an NTU game such that the boundary of each $V(S)$ is a hyperplane in $\mathbb{R}^S$. A *bargaining problem* is an NTU game where for all $S\subset N$, $V(S)\subseteq\Pi_{i\in S}V(\{i\})-\mathbb{R}^S_+$.

Let Γ be a non-empty class of games. A *solution* on Γ is a mapping σ which associates with every $(N,V)\in\Gamma$ a (possibly empty) subset $\sigma(N,V)$ of $V(N)$ for every $(N,V)\in\Gamma$.

Let $\Pi^N=\{P\subseteq N: p=2\}$, which is the set of two-person coalitions in N.

Definition 2. Let (N,V) be a game, $x\in V(N)$, and $P\in\Pi^N$. The two-person Davis–Maschler *reduced game* of (N,V) with respect to P given x_{-P} is the pair $(P, V_{x,P})$, consisting of the set P and the correspondence $V_{x,P}$ that associates with every coalition $S\subseteq P$ a subset $V_{x,P}(S)$ of $\mathbb{R}^P$, where

$$V_{x,P}(\{i\}) = \left\{y_i\in\mathbb{R}^{\{i\}} | \exists Q\subseteq N\setminus P, \quad (y_i,x_Q)\in V(\{i\}\cup Q)\right\}$$

for each $i\in P$,

$$V_{x,P}(P) = \left\{y_P\in\mathbb{R}^P | (y_p,x_{-p})\in V(N)\right\}.$$

Thus, given a payoff profile x for the grand coalition N, the feasible set for the pair P in the Davis–Maschler reduced game is what remains of $V(N)$ after the players not in P are paid according to x. In addition, each player in P expects to be able to cooperate with any of the players not in P provided they are paid their components of x. This is the way each player in P finds his "threat utility" against the other player in P.

Definition 3. Let (N,V) be a game, $x\in\partial V(N)$, and $P\in\Pi^N$. The two-person *reduced hyperplane game* of (N,V) with respect to P given x_{-P} is the pair $(P, W_{x,P})$, consisting of the set P and the correspondence $W_{x,P}$ that associates with every coalition $S\subseteq P$ a subset $W_{x,P}(S)$ of $\mathbb{R}^P$, where

$$W_{x,P}(\{i\}) = V_{x,P}(\{i\})$$

for each $i\in P$, and

$$W_{x,P}(P) = \left\{(y_i,y_j)\in\mathbb{R}^P | g_i(x)(y_i-x_i) + g_j(x)(y_j-x_j)\leq 0\right\}.$$

Thus, the reduced hyperplane game has the same feasible sets for the two individuals as the Davis–Maschler reduced game, but it prescribes the hyperplane tangent to the frontier at x_P for P: both players make the fictitious assumption that utility is transferable at the rates prescribed by the gradient of the frontier at x. Note how reduced hyperplane games are only defined for efficient payoff profiles in the grand coalition.

The *surplus* of player i against player j at the payoff vector x is defined as follows:

$$s_{i,j}(x) = v_i\left(x_{-\{i,j\}}\right) - x_i,$$

where

$$v_i\left(x_{-\{i,j\}}\right) = \max V_{x,\{i,j\}}(\{i\}).$$

That is, the surplus of player i against player j at the payoff profile x is the difference between the highest utility that player i could get without cooperating with j (when paying other players for their resources at the rate prescribed by x) and the utility that i receives at x.

Definition 4. Let (N,V) be a game. The *prekernel* of (N,V) is:

$$\mathcal{P}(N,V) = \left\{x\in\partial V(N) | \quad g_i(x)s_{i,j}(x) - g_j(x)s_{j,i}(x) = 0 \quad \forall i,j\in N\right\}.$$

The prekernel of a game is the set of efficient payoff profiles where each player's surplus (when weighted by the corresponding partial derivative of g at x) against every other player is the same. This is the sense in which each prekernel payoff has the flavor of a "bilateral equilibrium". See Serrano (1997) for an elaboration of this point in a non-cooperative setup, and Peleg (1986), Moldovanu (1990) and Serrano and Shimomura (1998) for characterizations of the prekernel. Serrano (1997) also explains how the surplus equations of the prekernel are justified, in spite of the apparent interpersonal utility comparisons.

Definition 5. Let (N,V) be a game. The *average prekernel* of (N,V) is:

$$\mathcal{AP}(N,V) = \left\{ x \in \partial V(N) | \sum_{j \in N \setminus \{i\}} \left[g_i(x) s_{i,j}(x) - g_j(x) s_{j,i}(x) \right] = 0 \quad \forall i \in N \right\}.$$

The average prekernel was introduced in Orshan and Zarzuelo (2000). It consists of those efficient payoff profiles in which each player is in a bilateral equilibrium situation only in average: it is possible that player i's surplus against player j exceeds that of player j against i, but this is offset by exactly the opposite situation with the players other than j.

3. Axiomatic analysis

In this section we characterize the average prekernel by means of seven logically independent axioms. The results here are to be compared with the closely related results in Serrano and Shimomura (1998). A comparison with the axiomatic result in Orshan and Zarzuelo (2000) will also be provided.

Let Γ be a non-empty class of games, and σ a solution on Γ. Then, we define the following properties.

Non-emptiness. For each $(N,V) \in \Gamma$, $\sigma(N,V) \neq \emptyset$.

Pareto efficiency. For each $(N,V) \in \Gamma$, $\sigma(N,V) \subseteq \partial V(N)$.

Let (N,v) be a TU game, and i,j be two distinct players in N. Then i and j are *substitutes* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.

Equal treatment for TU games. If $(N,v) \in \Gamma$ is a TU game and i and j are substitutes in (N,v), $x \in \sigma(N,v)$ implies that $x_i = x_j$.

Let (N,V) be a game, $\alpha \in \mathbb{R}^N_{++}$, and $\beta \in \mathbb{R}^N$. For each coalition S, we define the function $\lambda_S^{\alpha,\beta}$ from $\mathbb{R}^S$ to itself by

$$\lambda_S^{\alpha,\beta}(x_S) = (\alpha_i x_i + \beta_i)_{i \in S}$$

for each $x_S \in \mathbb{R}^S$. We then define $\lambda^{\alpha,\beta}(V)$ as the correspondence that associates with every coalition S a set

$$\lambda^{\alpha,\beta}(V)(S) = \left\{ y_S \in \mathbb{R}^S | \exists x_S \in V(S), y_S = \lambda_S^{\alpha,\beta}(x_S) \right\}.$$

That is, these two definitions simply describe positive affine transformations of the utility scales.

Scale invariance. For each $(N,V) \in \Gamma$, for each $\alpha \in \mathbb{R}^N_{++}$ and each $\beta \in \mathbb{R}^N$, $\sigma(N, \lambda^{\alpha,\beta}(V)) = \lambda_N^{\alpha,\beta}(\sigma(N,V))$.

Local independence. For each (N,V), $(N,V')\in\Gamma$ and $x\in\sigma(N,V)$, if $x\in\partial V(N)\cap\partial V'(N)$, $V(S)=V'(S)$ for each $S\neq N$, and $\nabla g(x)$ is proportional to $\nabla g'(x)$ (where g and g' are representations for $V(N)$ and $V'(N)$, respectively), then $x\in\sigma(N,V')$.

Non-emptiness and Pareto efficiency are standard conditions. Scale invariance is justified if one has in mind von Neumann–Morgenstern payoffs. Equal treatment for TU games is a weakening of Nash's (1950) symmetry axiom, as it is imposed only when utility is transferable. On the other hand, local independence is stronger than Nash's (1950) "independence of irrelevant alternatives" (IIA) axiom. The original formulation of local independence (e.g., Nagahisa, 1991) says that "if at a commodity allocation all agents have a common marginal rate of substitution under preference profiles u and u', then the allocation should be chosen as a socially optimal outcome for u' whenever it is selected for u". Our version, already used in Serrano and Shimomura (1998), expresses essentially the same concept in the payoff space. One justification of local independence is based on informational efficiency: in order to pin down the solution, local information (about the gradient of the Pareto frontier) suffices. If the problem looks the same locally, the solution will not change. There is no "action at a distance" in the influence of the shape of the feasible set on the location of the solution (see Nash (1953, p. 138)).

Proposition 1. *Let $\Gamma^{\{i,j\}}$ be the class of two-person games $(\{i,j\},V)$ satisfying Assumptions (1)–(4). Then a solution on $\Gamma^{\{i,j\}}$ satisfies non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence if and only if it is $\mathcal{AP}$.*

Proof. The proof is identical to that of Proposition 1 in Serrano and Shimomura (1998) after one observes that, over the considered class of games, $\mathcal{AP}=\mathcal{P}$. □

Thus, over the class of two-player games, the average prekernel coincides with the prekernel. If the set $V(\{i,j\})$ is convex and we have a bargaining problem, the average prekernel consists of a unique payoff profile, the one prescribed by the Nash solution.

Next, we shall present the axioms that will be operative in multilateral settings. To facilitate the comparison with the prekernel, we also state the axioms used in Serrano and Shimomura (1998).

Bilateral consistency. For each $(N,V)\in\Gamma$ and each $x\in\sigma(N,V)$, one has that $(P,V_{x,P})\in\Gamma$ and $x_P\in\sigma(P,V_{x,P})$ for each $P\in\Pi^N$.

Bilateral consistency says that the solution should be invariant to projections to two-player games, provided players have the expectations embodied in the bilateral Davis–Maschler reduced game.[7]

Converse consistency. For each $(N,V)\in\Gamma$ and each $x\in\partial V(N)$, if $(P,V_{x,P})\in\Gamma$ and $x_P\in\sigma(P,V_{x,P})$ for each $P\in\Pi^N$, then $x\in\sigma(N,V)$.

For a solution satisfying converse consistency, in order to impose its recommendations on a society, an arbitrator should simply make sure that the restrictions of the recommended payoff vector to each pair of players agrees with the solution (when the feasible possibilities for the players in the pair are described by the relevant reduced game).

[7] A strengthening of this property is consistency. A solution σ on a class Γ satisfies consistency if the same condition as above holds, but for all subsets P, i.e., not only restricted to those subsets P of N of cardinality 2 (see Thomson, 1996).

Serrano and Shimomura (1998) show that, over the class of smooth games, the prekernel $\mathcal{P}$ is the only solution satisfying bilateral consistency, converse consistency, and the five axioms for two-player games used in Proposition 1.[8]

We next formulate concepts of average consistency.

Average bilateral consistency. For each $(N,V)\in\Gamma$ and each $x\in\sigma(N,V)$, one has that $(P,V_{x,P})\in\Gamma$ for each $P\in\Pi^N$, and $x=\Sigma_{P\in\Pi^N}\frac{2}{n(n-1)}(y_P,x_{-p})$ for some $y_P\in\sigma(P,\ V_{x,P})$.

This is weaker than bilateral consistency: given a payoff profile x in the solution, it is required only that x be the average of $n(n-1)/2$ vectors: each of these, fixing the payoffs x_{-P}, consists of solution payoffs in each bilateral reduced game $(P,V_{x,P})$. If one thinks of the payoffs in the solution σ as being imposed by an arbitrator, suppose that $x\in\sigma(N,V)$ is what will be imposed. Following the logic of bilateral consistency, suppose that only pairs of players can appeal to the arbitrator and resubmit the bilateral problem for a new ruling (we assume that the possibilities of each pair are well described by their Davis–Maschler reduced game given x). Take an ex-ante perspective, so that it is not clear which pair of players will resubmit the problem to the arbitrator, and suppose that each pair is equally likely. If the solution satisfies average bilateral consistency, there exist expectations compatible with the use of the solution in each reduced game that would make each player i accept the payoff x_i. The reader will note that if the solution were bilaterally consistent, instead, this would be true no matter what the probability of each pair of players is, because the support of i's beliefs consists exclusively of projections of x, a much stronger property. Note also how, in the definition of average bilateral consistency, single-valuedness of the solution in reduced games is not required.

Average converse consistency. For each $(N,V)\in\Gamma$, if $(P,V_{x,P})\in\Gamma$ for each $P\in\Pi^N$ and $x=\Sigma_{P\in\Pi^N}\frac{2}{n(n-1)}(y_P,x_{-p})$ for some $y_P\in\sigma(P,V_{x,P})$, then $x\in\sigma(N,V)$.

Contrary to the relationship between bilateral consistency and its average counterpart, average converse consistency is stronger than converse consistency. First, it is not required that x be efficient. In addition, the requirement for the decentralization of x is now much weaker: we require from a payoff x that, if x is the average of solution/projection payoffs over all the associated bilateral reduced games, x must be recommended to the whole society as a solution point. That is, it suffices for x to be the expectation, over all equally likely pairs P, of vectors $(y_P,\ x_{-P})$ compatible with the use of σ in each reduced game, for the arbitrator to impose the payoff profile x. In contrast, converse consistency imposes x only if the projection x_P is in the solution for each pair P.

A class of games is *rich* if it includes games of more than two players and it contains all two-person games.

In cooperative game theory, Pareto efficiency is usually required as a desirable axiom. We study now the implications of Pareto efficiency, the other four axioms in Proposition 1 for two-player games, and average bilateral consistency and average converse consistency in our next impossibility result.

Theorem 1. *Let Γ_0 be a rich class of games satisfying Assumptions (1)–(4) that includes all games (N,V) in which V(N) is a convex set whose Pareto frontier does not contain flat segments. There is no solution on Γ_0 satisfying Pareto efficiency, average bilateral consistency and average*

[8] The results in Serrano and Shimomura (1998) are robust to a stronger version of converse consistency that replaces "$x\in\partial V(N)$" with "$x\in V(N)$". This observation is of interest for the sequel.

converse consistency, as well as the following four axioms for two player games: non-emptiness, equal treatment for TU games, scale invariance and local independence.

Proof. For $n=2$, average bilateral consistency reduces to bilateral consistency and average converse consistency to converse consistency. Therefore, by Proposition 1, the only solution satisfying the seven axioms considered is $\mathcal{P}=\mathcal{AP}$ over this class.

Let $n \geq 3$ and consider an NTU game (N,V) in which $V(N)$ is a convex set such that $\partial V(N)$ does not contain flat segments.[9] Suppose a solution σ satisfies all seven axioms. Let $x \in \sigma(N,V)$. By Pareto efficiency and average bilateral consistency of σ, and by our choice of the game excluding flat segments in $\partial V(N)$, we have that σ must satisfy bilateral consistency. That is, for all $P \in \Pi^N$, $x_P \in \sigma(P,V_{x,P})$. Since for all two-player games, we know that σ coincides with the prekernel, we have that for all $P \in \Pi^N, x_p \in \mathcal{P}(P,V_{x,P})$. By the converse consistency of the prekernel, we get that $x \in \mathcal{P}(N,V)$. Therefore, for every game (N,V) where $V(N)$ is a convex set for which $\partial V(N)$ does not include flat segments, $\sigma(N,V) \subseteq \mathcal{P}(N,V)$.

Now consider $x \in \mathcal{P}(N,V)$. By the bilateral consistency of the prekernel, we know that for all $P \in \Pi^N$, $x_p \in \mathcal{P}(P,V_{x,P})$, and given that $\sigma=\mathcal{P}$ for two-player games, we get that for all $P \in \Pi^N$, $x_P \in \sigma(P,V_{x,P})$. Therefore, by the average converse consistency of σ, $x \in \sigma(N,V)$. Hence, $\mathcal{P}(N,V) \subseteq \sigma(N,V)$.

Thus, over the class of games (N,V) where $V(N)$ is a convex set for which $\partial V(N)$ does not include flat segments, the only possibility is that σ is the prekernel. However, as shown by the following example, $\mathcal{P}$ does not satisfy average converse consistency on a game in this class: let $N=\{1, 2, 3\}$, $V(\{i\})$ is the non-positive real half-line for each $i \in N$, and $V(N)$ is the set whose Pareto frontier is a curved concave surface passing through the points $z^1=(1/3, 11/36, 13/36)$, $z^2=(13/36, 1/3, 11/36)$ and $z^3=(11/36, 13/36, 1/3)$. We impose also that at each of these z^h, for the two players i and j who do not receive a payoff of 1/3, $g_i(z^h)=g_j(z^h)$. As for the two-player coalitions, denote by cch the comprehensive hull of the convex hull of the given set. (To fit this example within all our assumptions on NTU games, the reader can consider a smooth and strictly comprehensive version of the frontier of the two-player coalitions outside of the relevant range. This will make no difference.) Then:

$$V(\{1,2\}) = \{(x_1,x_2) : (x_1,x_2) \in \text{cch}\{(1,0), (0,2/3)\}\};$$

$$V(\{1,3\}) = \{(x_1,x_3) : (x_1,x_3) \in \text{cch}\{(2/3,0), (0,1)\}\};$$

$$V(\{2,3\}) = \{(x_2,x_3) : (x_2,x_3) \in \text{cch}\{(1,0), (0,2/3)\}\}.$$

Let $x=(1/3, 1/3, 1/3)$. Note how, by local independence, $(11/36,13/36) \in \mathcal{P}(\{1,2\}, V_{x,\{1,2\}})$, $(13/36,11/36) \in \mathcal{P}(\{1,3\}, V_{x,\{1,3\}})$, and $(11/36,13/36) \in \mathcal{P}(\{2,3\}, V_{x,\{2,3\}})$. Therefore, the vector $x=(1/3, 1/3, 1/3)$, which is obtained as the average of z^1, z^2 and z^3, ought to be in $\mathcal{P}(N,V)$ if the prekernel satisfied average converse consistency. However, $x \notin \mathcal{P}(N,V)$. □

Remark 1. An impossibility result can be also obtained on any rich class of games by using all the same axioms and requiring local independence for n-player games. To see this, pick $x \in \sigma(N,V)$. By efficiency, $x \in \partial V(N)$. By local independence, if necessary, construct a game (N,U) with a strictly concave frontier and such that $x \in \sigma(N,U)$. After this change, the steps of the proof are identical as those in the Proof of Theorem 1.

[9] The next steps in this proof and the bulk of the Proof of Theorem 2 are familiar in theorems that use consistency. They comprise what Thomson (1996) calls the “elevator lemma”.

Remark 2. The Proof of Theorem 1 shows that the prekernel does not satisfy average converse consistency. For completeness, let us note that the average prekernel does not satisfy it either (as the same example shows). In addition, while the prekernel satisfies bilateral consistency, and hence its average counterpart, the average prekernel does not even satisfy average bilateral consistency. To see this, just use the same example as in the last step of the Proof of Theorem 1, and construct a symmetric Pareto frontier of $V(N)$. Then, the point $x=(\lambda, \lambda, \lambda)\in\partial V(N)$ will be in the average prekernel. However, this point cannot be obtained as an average over two-person coalitions P of points (y_P, λ) satisfying that $y_P\in\mathcal{AP}(P,V_{x,P})$.

As implied by one of the steps in the Proof of Theorem 1, efficiency and average consistency imply consistency when one considers NTU games. This fact and considerations similar to those in Remark 2 suggest the following alternative definitions.

Average bilateral consistency with respect to reduced hyperplane games. For each $(N,V)\in\Gamma$ and each $x\in\sigma(N,V)$, one has that $(P,W_{x,P})\in\Gamma$ for each $P\in\Pi^N$, and $x = \Sigma_{P\in\Pi^N}\frac{2}{n(n-1)}(y_P,x_{-P})$ for some $y_P\in\sigma(P,W_{x,P})$.

That is, much in the spirit of Shapley's (1969) λ-transfer principle, using as utility weights the gradient of $\partial V(N)$ at x, we can make the thought experiment that utility is transferable at those rates. Then, this version of average bilateral consistency has the same interpretation as the previous one: each payoff in the solution is the expectation of solution points to bilateral games where the threat points are determined by the Davis–Maschler logic, but where the feasible set is the half-space generated by the λ-transfer utility weights. See also Maschler and Owen (1989) and Orshan and Zarzuelo (2000) for two papers in which hyperplane games play a prominent role.

A similar comment applies to our next axiom. That is, we take the concept of average converse consistency, and combine it with the use of the λ-transfer thought experiment.

Average converse consistency with respect to reduced hyperplane games. For each $(N,V)\in\Gamma$, if $(P,W_{x,P})\in\Gamma$ for each $P\in\Pi^N$ and $x = \Sigma_{P\in\Pi^N}\frac{2}{n(n-1)}(y_P,x_{-P})$ for some $y_P\in\sigma(P,W_{x,P})$, then $x\in\sigma(N,V)$.

Remark 3. On a rich class of games, $\mathcal{AP}$ satisfies average bilateral consistency with respect to reduced hyperplane games and average converse consistency with respect to reduced hyperplane games.[10]

Our main result in this section follows:

Theorem 2. *Let Γ_0 be a rich class of games satisfying Assumptions (1)–(4). A solution on Γ_0 satisfies average bilateral consistency with respect to reduced hyperplane games, average converse consistency with respect to reduced hyperplane games, and the following five axioms for two-player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence) if and only if it is $\mathcal{AP}$.*

Proof. By Proposition 1 and Remark 3, the solution $\mathcal{AP}$ on Γ_0 satisfies the seven axioms listed. Now we prove uniqueness.

Let $(N,V)\in\Gamma_0$, and let σ be a solution on Γ_0 that also satisfies the seven axioms of the theorem. We prove that $\sigma(N,V) = \mathcal{AP}(N,V)$. We have already proven the case of $n=2$ (Proposition 1 and Remark 3). Then consider the case of $n\geq 3$.

[10] As additional support to these two axioms, we note that the Pareto efficiency correspondence and the core satisfy them over rich classes of games.

Let $x \in \mathcal{AP}(N,V)$. By the average bilateral consistency with respect to reduced hyperplane games of $\mathcal{AP}$, there exist $n(n-1)/2$ vectors (y_P, x_{-P}), $y_P \in \mathcal{AP}(P, W_{x,P})$ for all $P \in \Pi^N$ such that $x = \Sigma_{P \in \Pi^N} \frac{2}{n(n-1)}(y_p, x_{-p})$. But for every two-player game $(P, W_{x,P})$, if σ satisfies all seven axioms, $\sigma(P, W_{x,P}) = \mathcal{AP}(P, W_{x,P})$. Hence, there exist $n(n-1)/2$ vectors (y_P, x_{-P}), $y_P \in \sigma(P, W_{x,P})$ for all $P \in \Pi^N$ such that $x = \Sigma_{P \in \Pi^N} \frac{2}{n(n-1)}(y_p, x_{-p})$. By the average converse consistency with respect to reduced hyperplane games of σ, $x \in \sigma(N,V)$. Hence, $\mathcal{AP}(N,V) \subseteq \sigma(N,V)$.

To prove the opposite inclusion, we follow analogous steps. Take $x \in \sigma(N,V)$. By average bilateral consistency with respect to reduced hyperplane games of σ, for each $P \in \Pi^N$, $(P, W_{x,P}) \in \Gamma_0$ and x can be obtained as the average of vectors y_P, x_{-P} for some $y_P \in \sigma(P, W_{x,P})$. Since over the considered class of two-player games, $\sigma = \mathcal{AP}$, we have that for each $P \in \Pi^N$, $y_P \in \mathcal{AP}\ (P, W_{x,P})$, and by the average converse consistency with respect to reduced hyperplane games of the average prekernel, $x \in \mathcal{AP}(N,V)$. This establishes that $\sigma(N,V) \subseteq \mathcal{AP}(N,V)$. Therefore, $\sigma(N,V) = \mathcal{AP}(N,V)$. □

Remark 4. The basis of the characterization in Theorem 2 is to pin down the solution on two-person games with the five axioms imposed there, and to extend it to multilateral settings with the versions of average consistency and its converse for reduced hyperplane games. This was also the methodology in Serrano and Shimomura (1998), except that there bilateral consistency and its converse were used, instead of their average versions for reduced hyperplane games. In contrast, the axiomatic result in Orshan and Zarzuelo (2000) is based on characterizing the solution first on the class of hyperplane games. They then require their versions of consistency and its converse only over this class and use local independence for the extension of the result to general smooth NTU games. Note, for example, how their version of consistency (2-CO) is violated by $\mathcal{AP}$ outside of the class of hyperplane games because it is not generally single-valued.

Remark 5. Theorem 2 can also be stated over classes of games that are not rich. For example, the characterization would extend over the class of games where feasible sets are convex.

Next we show that the seven axioms used in the characterization are logically independent. In each example, the axiom in brackets is the one violated by the solution proposed.

Example 1 [Non-emptiness]. For every $(N,V) \in \Gamma_0$, let $\sigma(N,V) = \emptyset$. Then σ vacuously satisfies all the conditions of Theorem 2 except non-emptiness for two-person games.

Example 2 [Pareto efficiency]. For every two-person game (P,V), define $b(P, V) = (v_i)_{i \in P}$ if $(v_i)_{i \in P} \in \text{Int}V(P)$ and $b(P,V) = \mathcal{AP}(P,V)$ otherwise. For every $(N,V) \in \Gamma_0$, let

$$\sigma(N,V) = \left\{ x \in V(N) | x = \sum_{P \in \Pi^N} \frac{2}{n(n-1)} (y_P, x_{-P}) \qquad \text{for some } y_P \in b(P, W_{x,P}) \right\}.$$

Then σ violates Pareto efficiency for two-person games whenever $(v_i)_{i \in P}$ is in the interior of $V(P)$. Clearly, it satisfies non-emptiness, equal treatment for TU games, scale invariance and local independence over two-person games. By construction, σ also satisfies average bilateral consistency with respect to reduced hyperplane games and average converse consistency with respect to reduced hyperplane games.

Example 3 [Equal treatment for TU games]. For every $(N,V) \in \Gamma_0$, let

$$\sigma(N,V) = \partial V(N).$$

Then σ satisfies all the conditions of Theorem 2 except equal treatment for two-person TU games. That σ satisfies all the other axioms for two-person games is obvious. In addition, σ satisfies bilateral consistency with respect to reduced hyperplane games, and hence, also its weaker average version. Finally, it also satisfies average converse consistency with respect to reduced hyperplane games because these reduced games are defined only for efficient payoff profiles.

Example 4 [Scale invariance]. For every $(N,V)\in\Gamma_0$, let

$$\sigma(N,V) = \left\{x\in\partial V(N)|\sum_{j\in N\setminus\{i\}}[s_{i,j}(x)-s_{j,i}(x)] = 0 \quad \forall i\in N\right\}.$$

This is a version of the average prekernel, except that the surpluses are not weighted by the value of the partial derivatives of g at the Pareto frontier of $V(N)$. Then it is easy to see that σ violates scale invariance for two-person games, while it clearly satisfies all the other axioms for two-person games. Since the solution is based on a certain average surplus for each player, it satisfies the two average consistency axioms with respect to reduced hyperplane games.

Example 5 [Local independence]. For every two-person game (P,V), define $a(P,V)=(a_i(P,V))_{i\in P}$ by

$$a_i(P,V) = \max\left\{x_i\in\mathbb{R}^{\{i\}}|(x_i,v_j(x_{-P}))\in V(P), \quad \text{where} \quad P=\{i,j\}\right\}.$$

For every $(N,V)\in\Gamma_0$, let

$$\sigma(N,V) = \left\{x\in\partial V(N)|\ x = \sum_{P\in\Pi^N}\frac{2}{n(n-1)}(y_P,x_{-P})\right\}$$

for each $P=\{i,j\}$ and for some

$$y_P = (y_i,y_j)\in\partial W_{\{x\},\{i,j\}}(\{i,j\})\cap\left[(v_k(x_{-\{i,j\}}))_{k=i,j},a(\{i,j\},W_{x,\{i,j\}})\right],$$

where $[c, d]=\{(1-t)c+td|0\leq t\leq 1\}$ for each $c,d\in\mathcal{R}^{\{i,j\}}$. That is, for every player $i\in N$, x_i can be expressed as the average of maximal points of the feasible set $W_{x,\{i,j\}}(\{i, j\})$ on the segment connecting $(v_k(x_{-\{i,j\}}))_{k=i,j}$ to $a(\{i,j\}, W_{x,\{i,j\}})$ if $x\in\sigma(N,V)$. Note that σ is a sort of average of Kalai and Smorodinsky's (1975) bargaining solutions. Thus, it satisfies all the conditions of Theorem 2 for two-person games except local independence. By construction, it also satisfies the two average consistency axioms with respect to reduced hyperplane games.

Example 6 [Average bilateral consistency with respect to reduced hyperplane games]. Let $(N,V)\in\Gamma_0$. Let $\sigma(N,V) = \mathcal{AP}(N,V)$ when $n=2$, and $\sigma(N,V)=\partial V(N)$ for $n\geq 3$. By Proposition 1, σ satisfies the five axioms imposed on two-person games. Further, it is easy to see that average converse consistency with respect to reduced hyperplane games is also satisfied (see the argument in Example 3). To see that σ violates average bilateral consistency with respect to reduced hyperplane games, consider the three-player TU game in which $v(N)=1$ and $v(S)=0$ for all other S, and take $x=(0.7, 0.2, 0.1)\in\sigma(N,v)$. Note that x is not the average of the three vectors (y_P,x_{-P}) for $y_P\in\sigma(P,W_{x,P})$ for each two-player reduced game, i.e., (0.45, 0.45, 0.1), (0.4, 0.2, 0.4) and (0.7, 0.15, 0.15).

Example 7 [Average converse consistency with respect to reduced hyperplane games]. For every $(N,V)\in\Gamma_0$, let $\sigma(N,V)=\mathcal{P}(N,V)$. By Proposition 1, σ satisfies all the axioms for two-player games. Moreover, $\mathcal{P}$ satisfies bilateral consistency, and therefore, average bilateral consistency (with respect to Davis–Maschler reduced games and with respect to reduced hyperplane games). To see that $\mathcal{P}$ violates average converse consistency with respect to reduced hyperplane games, consider the same game at the end of the Proof of Theorem 1, but where $\partial V(N)$ is the hyperplane of equation $x_1+x_2+x_3=1$. Then, $x=(1/3,\ 1/3,\ 1/3)$ is the average of the three vectors $z^h=(y_P,\ x_{-P})$ with $y_P\in\mathcal{P}(P,W_{x,P})$ for each $P\in\Pi^N$. However, $x\notin\mathcal{P}(N,V)$.

As in Serrano and Shimomura (1998), we next investigate the implications of the same sets of seven axioms on the class of games with non-empty cores.

Definition 6. Let (N,V) be a game, S a non-empty subset of N, and $x\in\mathbb{R}^N$. Then we say that S can *improve upon* x if there is $y\in V(S)$ such that $y_i>x_i$ for all $i\in S$. The *core* of (N,V) is:

$$\mathcal{C}(N,V)=\{x\in V(N)|\quad \text{There is no coalition that can improve upon } x\}.$$

Definition 7. Let (N,V) be a game. The *intersection of the core and the prekernel* of (N,V) is:

$$\mathcal{P}^*(N,V)=\mathcal{C}(N,V)\cap\mathcal{P}(N,V).$$

Definition 8. Let (N,V) be a game. The *intersection of the core and the average prekernel* of (N,V) is:

$$\mathcal{AP}^*(N,V)=\mathcal{C}(N,V)\cap\mathcal{AP}(N,V).$$

We can show the following in exactly the same way as for Proposition 1.

Proposition 2. Let $\Gamma_c^{\{i,j\}}$ be the class of two-person games $(\{i,j\},V)$ satisfying Assumptions (1)–(4) and with non-empty cores. Then a solution on $\Gamma_c^{\{i,j\}}$ satisfies non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence if and only if it is $\mathcal{AP}^*$.

The proof simply follows from the fact that for two-player games $\mathcal{P}^*=\mathcal{AP}^*$. Serrano and Shimomura (1998) show that, over a rich class of smooth n-player games, $\mathcal{P}^*$ is the only solution satisfying bilateral consistency, converse consistency, and the group of five axioms for two-player games in Proposition 2. Similarly, using the following remark, we can prove the theorems below as we have done for Theorems 1 and 2.

Remark 6. On a class of games satisfying Assumptions (1)–(4) with non-empty cores containing all the two-person games with the same properties, $\mathcal{AP}^*$ satisfies average bilateral consistency with respect to reduced hyperplane games and average converse consistency with respect to reduced hyperplane games, because so do $\mathcal{AP}$ and $\mathcal{C}$.

Theorem 3. *Let Γ_c be the class of games satisfying Assumptions (1)–(4) with non-empty cores, containing all the two-person games with the same properties and also containing all games (N,V) with the same properties in which V(N) is a convex set whose Pareto frontier does not contain flat segments. There is no solution on Γ_c satisfying Pareto efficiency, average bilateral consistency and average converse consistency, as well as the following four axioms for two-player games: non-emptiness, equal treatment for TU games, scale invariance and local independence.*

Theorem 4. *Let Γ_c be the class of games satisfying Assumptions (1)–(4) with non-empty cores containing all the two-person games with the same properties. A solution on Γ_c satisfies average bilateral consistency with respect to reduced hyperplane games, average converse consistency with respect to reduced hyperplane games, and the following five axioms for two-player games (non-emptiness, Pareto efficiency, equal treatment for TU games, scale invariance and local independence) if and only if it is $\mathcal{AP}$*.*

It can be shown by modifying the seven examples that followed the Proof of Theorem 2 that the same seven axioms are also logically independent over the smaller class of games Γ_c.

4. Conclusion

We have obtained axiomatic foundations of the average prekernel, and we have compared them to those for the prekernel. The prekernel captures naturally those payoff profiles where players find themselves in a situation of "bilateral equilibrium". Given its frequent existence difficulties, it is convenient to settle for the weaker property of "bilateral equilibrium in average" that the average prekernel describes. It will also be interesting to test the differences between the two solution concepts in specific applications.

We close with a comment on the strategic foundations of the average prekernel. In the non-cooperative analysis performed in Serrano and Shimomura (2005), given a status quo payoff, a player is chosen at random every period and asked whether he accepts or rejects the status quo. If he accepts, the status quo is unaltered and a new player will be randomly selected the next period and asked the same question. If he rejects, he will bargain with another player for a redistribution of payoffs, but at the time he has to respond he does not know who will be his opponent in the bilateral bargaining round. In this round, upon rejection of a proposal, the status quo prevails with high probability, while with the rest of probability the rejector has the option of hiring a coalition that includes him but excludes the proposer. Whatever the outcome of this bilateral bargaining round, a new status quo is so determined, followed by the random choice of a new player the next period, and so on. In Serrano and Shimomura (2005), we show that the stationary equilibrium payoffs of this model approximate the average prekernel payoffs as the probability of cooperating with coalitions vanishes. In contrast, the model in Serrano (1997) that yields the prekernel is such that every period a pair of players is chosen at random to bargain if they want to modify the status quo. Thus, while the rules of negotiation in Serrano (1997) lead to a situation of bilateral equilibrium for every pair, a story of "equilibrium in average" is told in Serrano and Shimomura (2005). One important drawback of the result in the latter paper, though, is that we were not able to identify sufficient conditions of general interest under which equilibrium sequences converge in the non-cooperative procedure. Therefore, the search for a better strategic procedure that justifies the average prekernel is still open.

Acknowledgements

We thank three anonymous referees, Robert Aumann, Michael Maschler, William Thomson, Federico Valenciano, Oscar Volij and Jośe Zarzuelo for useful comments. Serrano thanks Universidad Carlos III and CEMFI in Madrid for their hospitality, and acknowledges grants from Fundaci´ on Banco Herrero, Universidad Carlos III and the Alfred P. Sloan Foundation. Shimomura acknowledges research support from the COE research fund 09CE2002, Grant-in-Aid for Scientific Research A(2) 16203020, and the Zengin Foundation for Studies on Economics and Finance.

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