COOPERATIVE GAMES: CORE AND SHAPLEY VALUE

by

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Article Outline

Glossary:

Game theory: discipline that studies strategic situations.

Cooperative game: strategic situation involving coalitions, whose formation assumes the existence of binding agreements among players.

Characteristic or coalitional function: the most usual way to represent a cooperative game.

Solution concept: mapping that assigns predictions to each game.

Core: solution concept that assigns the set of payoffs that cannot be improved upon by any coalition.

Shapley value: solution concept that assigns the average of marginal contributions to coalitions.

I. Definition.

Cooperative game theory. It is one of the two counterparts of game theory. It studies the interactions among coalitions of players. Its main question is this: Given the sets of feasible payoffs for each coalition, what payoff will be awarded to each player? One can take a positive or normative approach to answering this question, and different solution concepts in the theory lean towards one or the other.

Core. It is a solution concept that assigns to each cooperative game the set of payoffs that no coalition can improve upon or block. In a context in which there is unfettered coalitional interaction, the core arises as a good positive answer to the question posed in cooperative game theory. In other words, if a payoff does not belong to the core, one should not expect to see it as the prediction of the theory if there is full cooperation.

Shapley value. It is a solution that prescribes a single payoff for each player, which is the average of all marginal contributions of that player to each coalition he or she is a member of. It is usually viewed as a good normative answer to the question posed in cooperative game theory. That is, those who contribute more to the groups that include them should be paid more.

Although there were som earlier contributions, the official date of birth of game theory is usually taken to be 1944, year of publication of the first edition of the *Theory of Games and Economic Behavior*, by John von Neumann and Oskar Morgenstern [42]. The core was first proposed by Francis Ysidro Edgeworth in 1881 [13], and later reinvented and defined in game theoretic terms in [14]. The Shapley value was proposed by Lloyd Shapley in his 1953 PhD dissertation [37]. Both the core and the Shapley value have been applied widely, to shed light on problems in different disciplines, including economics and political science.

II. Introduction.

Game theory is the study of games, also called strategic situations. These are decision problems with multiple decision makers, whose decisions impact one another. It is divided into two branches: non-cooperative game theory and cooperative game theory. The actors in non-cooperative game theory are individual players, who may reach agreements only if they are self-enforcing. The non-cooperative approach provides a rich language and develops useful tools to analyze games. One clear advantage of the approach is that it is able to model how specific details of the interaction among individual players may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt, and it is the subject of this article.

The actors in cooperative game theory are coalitions, that is, groups of players. For the most part, two facts, that coalitions can form and that each coalition has a feasible set of payoffs available to its members, are taken as given. Given the coalitions and their sets of feasible payoffs as primitives, the question tackled is the identification of final payoffs awarded to each player. That is, given a collection of feasible sets of payoffs, one for each coalition, can one predict or recommend a payoff (or set of payoffs) to be awarded to each player? Such predictions or recommendations are embodied in different solution concepts.

Indeed, one can take several approaches to answering the question just posed. From a positive or descriptive point of view, one may want to get a prediction of the likely outcome of the interaction among the players, and hence, the resulting payoff be understood as the natural consequence of the forces at work in the system. Alternatively, one can take a normative or prescriptive approach, set up a number of normative goals, typically embodied in axioms, and try to derive their logical implications. Although authors sometimes disagree on the classification of the different solution concepts according to these two criteria –as we shall see, the understanding of each solution concept is enhanced if one can view it from very distinct approaches–, in this article we shall exemplify the positive approach with the core and the normative approach with the Shapley value. While this may oversimplify the issues, it should be helpful to a reader new to the subject.

The rest of the article is organized as follows. Section III introduces the basic model of a cooperative game, and discusses its assumptions as well as the notion of solution concepts. Section IV is devoted to the core, and Section V to the Shapley value. In each case, some of the main results for each of the two are described, and examples are provided. Section VI discusses some directions for future research.

III. Cooperative Games.

III.a. Representations of Games. The Characteristic Function. Let us begin by presenting the different ways to describe a game. The first two are the usual ways employed in non-cooperative game theory.

The most informative way to describe a game is called its *extensive form.* It consists of a game tree, specifying the timing of moves for each player and the information available to each of them at the time of making a move. At the end of each path of moves, a final outcome is reached and a payoff vector is specified. For each player, one can define a *strategy*, i.e., a complete contingent plan of action to play the game. That is, a strategy is a function that specifies a feasible move each time a player is called upon to make a move in the game.

One can abstract from details of the interaction (such as timing of moves and information available at each move), and focus on the concept of strategies. That is, one can list down the set of strategies available to each player, and arrive at the *strategic* or *normal form* of the game. For two players, for example, the normal form is represented in a bimatrix table. One player controls the rows, and the other the columns. Each cell of the bimatrix is occupied with an ordered pair, specifying the payoff to each player if each of them chooses the strategy corresponding to that cell.

One can further abstract from the notion of strategies, which will lead to the *characteristic function form* of representing a game. From the strategic form, one makes assumptions about the strategies used by the complement of a coalition of players to determine the feasible payoffs for the coalition (see, for example, the derivations in [42] and [7]). This is the representation most often used in cooperative game theory.

Thus, here are the primitives of the basic model in cooperative game theory. Let $N = \{1, \ldots, n\}$ be a finite set of players. Each non-empty subset of N is called a *coalition*. The set N is referred to as the *grand coalition*. For each coalition S, we shall specify a set $V(S) \subset I\!R^{|S|}$ containing $|S|$-dimensional payoff vectors that are feasible for coalition S. This is called the characteristic function, and the pair (N, V) is called a *cooperative game*. Note how a reduced form approach is taken because one does not explain what strategic choices are behind each of the payoff vectors in $V(S)$. In addition, in this formulation, it is implicitly assumed that the actions taken by the complement coalition (those players in $N \setminus S$) cannot prevent S from achieving each of the payoff vectors in $V(S)$. There are more general models in which these sorts of externalities across coalitions are considered, but we shall ignore them in this article.

III.b. Assumptions on the Characteristic Function. Some of the most common technical assumptions made on the characteristic function are the following:

(1) For each $S \subseteq N$, $V(S)$ is closed. Denote by $\partial V(S)$ the boundary of $V(S)$. Hence, $\partial V(S) \subseteq V(S)$.

(2) For each $S \subseteq N$, $V(S)$ is comprehensive, i.e., for each $x \in V(S)$, $\{x\} - I\!R_+^{|S|} \subseteq V(S)$.

(3) For each $x \in I\!R^{|S|}$,

$$\partial V(S) \cap (\{x\} + I\!R_+^{|S|})$$

is bounded.

(4) For each $S \subseteq N$, there exists a continuously differentiable representation of $V(S)$, i.e., a continuously differentiable function $g_S : I\!R^{|S|} \to I\!R$ such that

$$V(S) = \{x \in I\!R^{|S|} | \; g_S(x) \leq 0\}.$$

(5) For each $S \subseteq N$, $V(S)$ is non-levelled, i.e., for every $x \in \partial V(S)$, the gradient of g_S at x is positive in all its coordinates.

With the assumptions made, $\partial V(S)$ is its Pareto frontier, i.e., the set of vectors $x_S \in V(S)$ such that there does not exist $y_S \in V(S)$ satisfying that $y_i \geq x_i$ for all $i \in S$ with at least one strict inequality.

Other assumptions usually made relate the possibilities available to different coalitions. Among them, a very important one is *balancedness*, which we shall define next:

A collection $\mathcal{T}$ of coalitions is balanced if there exists a set of weights $w(S) \in [0, 1]$ for each $S \in \mathcal{T}$ such that for every $i \in N$, $\sum_{S \in \mathcal{T}, S \supseteq \{i\}} w(S) = 1$. One can think of these weights as the fraction of time that each player devotes to each coalition he is a member of, with a given coalition representing the same fraction of time for each player. The game (N, V) is balanced if $x_N \in V(N)$ whenever $(x_S) \in V(S)$ for every S in a balanced collection $\mathcal{T}$. That is, the grand coalition can always implement any "time-sharing arrangement" that the different subcoalitions may come up with.

The characteristic function defined so far is often referred to as a *non-transferable utility (NTU)* game. A particular case is the *transferable utility (TU)* game case, in which for each coalition $S \subseteq N$, there exists a real number $v(S)$ such that

$$V(S) = \{x \in I\!R^{|S|} : \sum_{i \in S} x_i \leq v(S)\}.$$

Abusing notation slightly, we shall denote a TU game by (N, v). In the TU case there is an underlying nummeraire –money– that can transfer utility or payoff at a one-to-one rate from one player to any other. Technically, the theory of NTU games is far more complex: it uses convex analysis and fixed point theorems, whereas the TU theory is based on linear inequalities and combinatorics.

III.c. Solution Concepts. Given a characteristic function, i.e., a collection of sets $V(S)$, one for each S, the theory formulates its predictions on the basis of different *solution concepts*. We shall concentrate on the case in which the grand coalition forms, that is, cooperation is totally successful. Of course, solution concepts can be adapted to take care of the case in which this does not happen.

A *solution* is a mapping that assigns a set of payoff vectors in $V(N)$ to each characteristic function game (N, V). Thus, a solution in general prescribes a set, which can be empty, or a singleton (when it assigns a unique payoff vector as a function of the fundamentals of the problem). The leading set-valued cooperative solution concept is the core, while one of the most used single-valued ones is the Shapley value for TU games.

There are several criteria to evaluate the reasonableness or appeal of a cooperative solution. As outlined above, in a normative approach, one can propose axioms, abstract principles that one would like the solution to satisfy, and the next step is to pursue their logical consequences. Historically, this was the first argument to justify the Shapley value. Alternatively, one could start by defending a solution on the basis of its definition alone. In the case of the core, this will be especially natural: in a context in which players can freely get together in groups, the prediction should be payoff vectors that cannot be improved upon by any coalition. One can further enhance one's positive understanding of the solution concept by proposing games in extensive form or in normal form played non-cooperatively by players whose self-enforcing agreements lead to a given solution. This is simply to provide non-cooperative foundations or non-cooperative implementation to the cooperative solution in question, and it is an important research agenda initiated by John Nash in [25], referred to as the Nash program (see [34] for a recent survey). Today, there are interesting results of these different kinds for many solution concepts, which include axiomatic characterizations and non-cooperative foundations. Thus, one can evaluate the appeal of the axioms and the non-cooperative procedures behind each solution to defend a more normative or positive interpretation in each case.

IV. The Core.

The idea of agreements that are immune to coalitional deviations was first introduced to economic theory by Edgeworth in [13], which defined the set of coalitionally stable allocations of an economy under the name "final settlements." Edgeworth envisioned this concept as an alternative to *competitive equilibrium* [43], of central importance in economic theory, and was also the first to investigate the connections between the two concepts. Edgeworth's notion, which today we refer to as the *core*, was rediscovered and introduced to game theory in [14]. The origins of the core were not axiomatic. Rather, its simple and appealing definition appropriately describes stable outcomes in a context of unfettered coalitional interaction.

The core of the game (N, V) is the set of payoff vectors

$$C(N,V) = \{x \in V(N) : \nexists S \subseteq N, x_S \in V(S) \setminus \partial V(S)\}.$$

In words, it is the set of feasible payoff vectors for the grand coalition that no coalition can upset. If such a coalition S exists, we shall say that S can improve upon or block x, and x is deemed unstable. That is, in a context where any coalition can get together, when S has a blocking move, coalition S will form and abandon the grand coalition and its payoffs x_S in order to get to a better payoff for each of the members of the coalition, a plan that is feasible for them.

IV.a. Non-Emptiness. The core can prescribe the empty set in some games. A game with an empty core is to be understood as a situation of strong instability, as any payoffs proposed to the grand coalition are vulnerable to coalitional blocking.

Example: Consider the following simple majority 3-player TU game, in which the votes of at least two players makes the coalition winning. That is, we represent the situation by the following characteristic function: $v(S) = 1$ for any S containing at least two members, $v(\{i\}) = 0$ for all $i \in N$. Clearly, $C(N, v) = \emptyset$. Any feasible payoff agreement proposed to the grand coalition will be blocked by at least one coalition.

An important sufficient condition for the non-emptiness of the core of NTU games is balancedness, as shown in [32]:

Theorem (Scarf, [32]): Let the game (N, V) be balanced. Then $C(N, V) \neq \emptyset$.

For the TU case, balancedness is not only sufficient, but it becomes also necessary for the non-emptiness of the core:

Theorem (Bondareva, [9]; Shapley, [39]): Let (N, v) be a TU game. Then, (N, v) is balanced if and only if $C(N, v) \neq \emptyset$.

IV.b. The Connections with Competitive Equilibrium. In economics, the institution of markets and the notion of prices are essential to the understanding of the allocation of goods and the distribution of wealth among individuals. For simplicity in the presentation, we shall concentrate on exchange economies, and disregard production aspects. That is, we shall assume that the goods in question have already been produced in some fixed amounts, and now they are to be allocated to individuals to satisfy their consumption needs.

An *exchange economy* is a system in which each agent i in the set N has a consumption set $Z_i \subseteq \mathbb{R}^l_+$ of commodity bundles, as well as a preference relation over Z_i and an initial endowment $\omega_i \in Z_i$ of the commodities. A feasible *allocation* of goods in the economy is a list of bundles $(z_i)_{i \in N}$ such that $z_i \in Z_i$ and $\sum_{i \in N} z_i \leq \sum_{i \in N} \omega_i$. An allocation is *competitive* if it is supported by a *competitive equilibrium.* A competitive equilibrium is a price-allocation pair $(p, (z_i)_{i \in N})$, where $p \in \mathbb{R}^l \setminus \{0\}$ is such that

- for every $i \in N$, z_i is top-ranked for agent i among all bundles z satisfying that $pz \leq p\omega_i$,
- and $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$.

In words, this is what the concept expresses. First, at the equilibrium prices, each agent demands z_i, i.e., wishes to purchase this bundle among the set of affordable bundles, the budget set. And second, these demands are such that all markets clear, i.e., total demand equals total supply.

Note how the notion of a competitive equilibrium relies on the principle of private ownership (each individual owns his or her endowment, which allows him or her to access markets and purchase things). Moreover, each agent is a price-taker in all markets. That is, no single individual can affect the market prices with his or her actions; prices are fixed parameters in each individual's consumption decision. The usual justification for the price-taking assumption is that each individual is "very small" with respect to the size of the economy, and hence, has no market power.

One difficulty with the competitive equilibrium concept is that it does not explain where prices come from. There is no single agent in the model responsible for coming up with them. Walras in [43] told the story of an auctioneer calling out prices until demand and supply coincide, but in many real-world markets there is no auctioneer. More generally, economists attribute the equilibrium prices to the workings of the forces of demand and supply, but this appears to be simply repeating the definition. So, is there a different way one can explain competitive equilibrium prices?

As it turns out, there is a very robust result that answers this question. We refer to it as the *equivalence principle* (see, e.g., [6]), by which, under certain regularity conditions, the predictions provided by different game-theoretic solution concepts, when applied to an economy with a large enough set of agents, tend to converge to the set of competitive equilibrium allocations. One of the first results in this tradition was provided by Edgeworth in 1881 for the core. Note how the core of the economy can be defined in the space of allocations, using the same definition as above. Namely, a feasible allocation is in the core if it cannot be blocked by any coalition of agents when making use of the coalition's endowments.

Edgeworth's result was generalized later by Debreu and Scarf in [11] for the case in which an exchange economy is replicated an arbitrary number of times (Anderson studies in [1] the more general case of arbitrary sequences of economies, not necessarily replicas). An informal statement of the Debreu-Scarf theorem follows:

Theorem (Debreu and Scarf, [11]): Consider an exchange economy. Then,

(i) The set of competitive equilibrium allocations is contained in the core.

(ii) For each non-competitive core allocation of the original economy, there exists a sufficiently large replica of the economy for which the replica of the allocation is blocked.

The first part states a very appealing property of competitive allocations, i.e., their coalitional stability. The second part, known as the core convergence theorem, states that the core "shrinks" to the set of competitive allocations as the economy grows large.

In [3], Aumann models the economy as an atomless measure space, and demonstrates the following core equivalence theorem:

Theorem (Aumann, [3]): Let the economy consists of an atomless continuum of agents. Then, the core coincides with the set of competitive allocations.

For readers who wish to pursue the topic further, [2] provides a recent survey.

IV.c. Axiomatic Characterizations. The axiomatic foundations of the core were provided much later than the concept was proposed. These characterizations are all inspired by Peleg's work. They include [26], [27], and [36] – the latter paper also provides an axiomatization of competitive allocations in which core convergence insights are exploited.

In all these characterizations, the key axiom is that of *consistency*, also referred to as the reduced game property. Consistency means that the outcomes prescribed by a solution should be "invariant" to the number of players in the game. More formally, let (N, V) be a game, and let σ be a solution. Let $x \in \sigma(N, V)$. Then, the solution is consistent if for every $S \subseteq N$, $x_S \in \sigma(S, V_{xS})$, where (S, V_{xS}) is the reduced game for S given payoffs x, defined as follows. The feasible set for S in this reduced game is the projection of $V(N)$ at $x_{N\setminus S}$, i.e., what remains after paying those outside of S:

$$V_{xS}(S) = \{y_S : (y_S, x_{N\setminus S}) \in V(N)\}.$$

However, the feasible set of $T \subset S$, $T \neq S$, allows T to make deals with any coalition outside of S, provided that those services are paid at the rate prescribed by $x_{N\setminus S}$:

$$V_{xS}(T) = \{y_T \in \cup_{Q\subseteq N\setminus S}(y_T, x_Q) \in V(T \cup Q)\}.$$

It can be shown that the core satisfies consistency with respect to this reduced game. Moreover, consistency is the central axiom in the characterization of the core, which, depending on the version one looks at, uses a host of other axioms; see [26], [27], and [36].

IV.d. Non-Cooperative Implementation. To obtain a non-cooperative implementation of the core, the procedure must embody some feature of anonymity, since the core is usually a large set and it contains payoffs where different players are treated very differently. For instance, if the procedure always had a fixed set of moves, typically the prediction would favor the first mover, making it impossible to obtain an implementation of the entire set of payoffs.

The model in [30] builds in this anonymity by assuming that negotiations take place in continuous time, so that anyone can speak at the beginning of the game, and at any point in time, instead of having a fixed order. The player that gets to speak first makes a proposal consisting of naming a coalition that contains him and a feasible payoff for that coalition. Next, the players in that coalition get to respond. If they all accept the proposal, the coalition leaves and the game continues among the other players. Otherwise, a new proposal may come from any player in N. It is shown that, if the TU game has a non-empty core (as well as any of its subgames), a class of stationary self-enforcing predictions of this procedure coincide with the core. If a core payoff is proposed to the grand coalition, there are no incentives for individual players to reject it. Conversely, a non-core payoff cannot be sustained because any player in a blocking coalition has an incentive to make a proposal to that coalition, who will accept it (knowing that the alternative, given stationarity, would be to go back to the non-core status quo). [24] offers a discrete-time version of the mechanism: in this work, the anonymity required is imposed on the solution concept, by looking at the order-independent equilibria of the procedure.

The model in [33] sets up a market to implement the core. The anonymity of the procedure stems from the random choice of broker. The broker announces a vector $(x_1, \ldots, x_n)$, where the components add up to $v(N)$. One can interpret x_i as the price for the productive asset held by player i. Following an arbitrary order, the remaining players either accept or reject these prices. If player i accepts, he sells his asset to the broker for the price x_i and leaves the game. Those who reject get to buy from the broker, at the called out prices, the portfolio of assets of their choice if the broker still has them. If a player rejects, but does not get to buy the portfolio of assets he would like because someone else took them before, he can always leave the market with his own asset. The broker's payoff is the worth of the final portfolio of assets that he holds, plus the net monetary transfers that he has received. It is shown in [33] that the prices announced by the broker will always be his top-ranked vectors in the core. If the TU game is such that gains from cooperation increase with the size of coalitions, a beautiful theorem of Shapley in [41] is used to prove that the set of all equilibrium payoffs of this procedure will coincide with the core. Core payoffs are here understood as those price vectors where all arbitrage opportunities in the market have been wiped out. Also, procedures in [35] implement the core, but do not rely on the TU assumption, and they use a procedure in which the order of moves can be endogenously changed by players. Finally, yet another way to build anonymity in the procedure is by allowing the proposal to be made by brokers outside of the set N, as done in [28].

IV.e. An Application. Consider majority games within a parliament. Suppose there are 100 seats, and decisions are made by simple majority so that 51 votes are required to pass a piece of legislation.

In the first specification, suppose there is a very large party –player 1–, who has 90 seats. There are five small parties, with 2 seats each. Given the simple majority rules, this problem can be represented by the following TU characteristic function: $v(S) = 1$ if S contains player 1, and $v(S) = 0$ otherwise. The interpretation is that each winning coalition can get the entire surplus –pass the desired proposal. Here, a coalition is winning if and only if player 1 is in it. For this problem, the core is a singleton: the entire unit of surplus is allocated to player 1, who has all the power. Any split of the unit surplus of the grand coalition ($v(N) = 1$) that gives some positive fraction of surplus to any of the small parties can be blocked by the coalition of player 1 alone.

Consider now a second problem, in which player 1, who continues to be the large party, has 35 seats, and each of the other five parties has 13 seats. Now, the characteristic function is as follows: $v(S) = 1$ if and only if S either contains player 1 and two small parties, or it contains four of the small parties; $v(S) = 0$ otherwise. It is easy to see that now the core is empty: any split of the unit surplus will be blocked by at least one coalition. For example, the entire unit going to player 1 is blocked by the coalition of all five small parties, which can award 0.2 to each of them. But this arrangement, in which each small party gets 0.2 and player 1 nothing, is blocked as well, because player 1 can bribe two of the small parties (say, players 2 and 3) and promise them 1/3 each, keeping the other third for itself, and so on. The emptiness of the core is a way to describe the fragility of any agreement, due to the inherent instability of this coalition formation game.

V. The Shapley Value.

Now consider a transferable utility or TU game in characteristic function form. The number $v(S)$ is referred to as the worth of S, and it expresses S's initial position (e.g., the maximum total amount of surplus in nummeraire –money, or power– that S initially has at its disposal.

V.a. Axiomatics. Shapley in [37] is interested in solving in a fair and unique way the problem of distribution of surplus among the players, when taking into account the worth of each coalition. To do this, he restricts attention to single-valued solutions and resorts to the axiomatic method. He proposes the following axioms on a single-valued solution:

(i) Efficiency: The payoffs must add up to $v(N)$, which means that all the grand coalition surplus is allocated.

(ii) Symmetry: If two players are substitutes because they contribute the same to each coalition, the solution should treat them equally.

(iii) Additivity: The solution to the sum of two TU games must be the sum of what it awards to each of the two games.

(iv) Dummy player: If a player contributes nothing to every coalition, the solution should pay him nothing.

(To be precise, the name of the first axiom should be different. In an economic sense, the statement does imply efficiency in superadditive games, i.e., when for every pair of disjoint coalitions S and T, $v(S) + v(T) \leq v(S \cup T)$. In the absence of superadditivity, though, forming the grand coalition is not necessarily efficient, because a higher aggregate payoff can be obtained from a different coalition structure.)

The surprising result in [37] is this:

Theorem (Shapley, [37]): There is a unique single-valued solution to TU games satisfying efficiency, symmetry, additivity and dummy. It is what today we call the Shapley value, the function that assigns to each player i the payoff

$$\mathrm{Sh}_i(N, v) = \sum_{S, i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}[v(S) - v(S \setminus \{i\})].$$

That is, the Shapley value awards to each player the average of his marginal contributions to each coalition. In taking this average, all orders of the players are considered to be equally likely. Let us assume, also without loss of generality, that $v(\{i\}) = 0$ for each player i.

What is especially surprising in Shapley's result is that nothing in the axioms (with the possible exception of the dummy axiom) hints at the idea of marginal contributions, so marginality in general is the outcome of all the axioms, including additivity or linearity. Among the axioms utilized by Shapley, additivity is the one with a lower normative content: it is simply a mathematical property to justify simplicity in the computation of the solution. Young in [45] provides a beautiful counterpart to Shapley's theorem. He drops additivity (as well as the dummy player axiom), and instead, uses an axiom of marginality. Marginality means that the solution should pay the same to a pleyar in two games if his or her marginal contributions to coalitions is the same in both games. Marginality is an idea with a strong tradition in economic theory. Young's result is "dual" to Shapley's, in the sense that marginality is assumed and additivity derived as the result:

Theorem (Young, [45]): There exists a unique single-valued solution to TU games satisfying efficiency, symmetry and marginality. It is the Shapley value.

Apart from these two, [19] provides further axiomatizations of the Shapley value using the idea of potential and the concept of consistency, as described in the previous section.

There is no single way to extend the Shapley value to the class of NTU games. There are three main extensions that have been proposed: the Shapley λ-transfer value [40], the Harsanyi value [16], and the Maschler-Owen consistent value [23]. They were axiomatized in [5], [17], and [10], respectively.

V.b. The Connections with Competitive Equilibrium. As was the case for the core, there is a value equivalence theorem. The result holds for the TU domain (see [38], [4], [8]). It can be shown that the Shapley value payoffs can be supported by competitive prices. Furthermore, in large enough economies, the set of competitive payoffs "shrinks" to approximate the Shapley value. However, the result cannot be easily extended to the NTU domain. While it holds for the λ-transfer value, it need not obtain for the other extensions. For further details, the interested reader is referred to [18] and the references therein.

V.c. Non-Cooperative Implementation. Reference [15] was the first to propose a procedure that provided some non-cooperative foundations of the Shapley value. Later, other authors have provided alternative procedures and techniques to the same end, including [44], [21], [20], and [29].

We shall concentrate on the description of the procedure proposed by Hart and Mas-Colell in [20]. Generalizing an idea found in [22], which studies the case of $\delta = 0$ –see below–, Hart and Mas-Colell propose the following non-cooperative procedure. With equal probability, each player $i \in N$ is chosen to publicly make a feasible proposal to the others: $(x_1, \ldots, x_n)$ is such that the sum of its components cannot exceed $v(N)$. The other players get to respond to it in sequence, following a prespecified order. If all accept, the proposal is implemented; otherwise, a random device is triggered. With probability $0 \leq \delta < 1$, the same game continues being played among the same n players (and thus, a new proposer will be chosen again at random among them), but with probability $1 - \delta$, the proposer leaves the game. He is paid 0 and his resources are removed, so that in the next period, proposals to the remaining $n - 1$ players cannot add up to more than $v(N \setminus \{i\})$. A new proposer is chosen at random among the set $N \setminus \{i\}$, and so on.

As shown in [20], there exists a unique stationary self-enforcing prediction of this procedure, and it actually coincides with the Shapley value payoffs for any value of δ. (Stationarity means that strategies cannot be history dependent). As $\delta \to 1$, the Shapley value payoffs are also obtained not only in expectation, but with independence of who is the proposer. One way to understand this result, as done in [20], is to check that the rules of the procedure and stationary behavior in it are in agreement with

Shapley's axioms. That is, the equilibrium relies on immediate acceptances of proposals, stationary strategies treat substitute players similarly, the equations describing the equilibrium have an additive structure, and dummy players will have to receive 0 because no resources are destroyed if they are asked to leave. It is also worth stressing the important role in the procedure of players' marginal contributions to coalitions: following a rejection, a proposer incurs the risk of being thrown out and the others of losing his resources, which seem to suggest a "price" for them.

In [21], the authors study the conditions under which stationarity can be removed to obtain the result. Also, [29] uses a variant of the Hart and Mas-Colell procedure, by replacing the random choice of proposers with a bidding stage, in which players bid to obtain the right to make proposals.

V.d. An Application. Consider again the class of majority problems in a parliament consisting of 100 seats. As we shall see, the Shapley value is a good way to understand the power that each party has in the legislature.

Let us begin by considering again the problem in which player 1 has 90 seats, while each of the five small parties has 2 seats. It is easy to see that the Shapley value, like the core in this case, awards the entire unit of surplus to player 1: effectively, each of the small parties is a dummy player, and hence, the Shapley value awards zero to each of them.

Consider a second problem, in which player 1 is a big party with 35 seats, and there are 5 small parties, with 13 seats each. The Shapley value awards 1/3 to the large party, and, by symmetry, 2/15 to each of the small parties. To see this, we need to see when the marginal contributions of player 1 to any coalition are positive. Recall that there are 6! possible orders of players. Note how, if player 1 arrives first or second in the room in which the coalition is forming, his marginal contribution is zero: the coalition was losing before he arrived and continues to be a losing coalition after his arrival. Similarly, his marginal contribution is also zero if he arrives fifth or sixth to the coalition; indeed, in this case, before he arrives the coalition is already winning, so he adds nothing to it. Thus, only when he arrives third or fourth, which happens a third of the times, does he change the nature of the coalition, from losing to winning. This explains his Shapley value share of 1/3. In this game, the Shapley value payoffs roughly correspond to the proportion of seats that each party has.

Next, consider a third problem in which there are two large parties, while the other four parties are very small. For example, let each of the large parties have 48 seats (say, players 1 and 2), while each of the four small parties has only one seat. Now, the Shapley value payoffs are 0.3 to each of the two large parties, and 0.1 to each of the small ones. To see this, note that the marginal contribution of a small party is only positive when he comes fourth in line, and out of the preceding three parties in the coalition, exactly one of them is a large party, i.e., 72 orders out of the 5! orders in which he is fourth. That is, $(72/5!) \times (1/6) = 1/10$. In this case, the competition between the large parties for the votes of the small parties increases the power of the latter quite significantly, with respect to the proportion of seats that each of them holds.

Finally, consider a fourth problem with two large parties (players 1 and 2) with 46 seats each, one mid-size party (player 3) with 5 seats, and three small parties, each with one seat. First, note that each of the three small parties has become a dummy player: no winning coalition where he belongs becomes losing if he leaves the coalition, and so players 4, 5 and 6 are paid zero by the Shapley value. Now, note that, despite the substantial difference of seats between each large party and the mid-size party, each of them is identical in terms of marginal contributions to a winning coalition. Indeed, for $i = 1, 2, 3$, player i's marginal contribution to a coalition is positive only if he arrives second or third or fourth or fifth (and out of the preceding players in the coalition, exactly one is one of the non-dummy players). Note how the Shapley value captures nicely the changes in the allocation of power due to each different political scenario. In this case, the fierce competition between the large parties for the votes of player 3, the swinging party to form a majority, explains the equal share of power among the three.

VI. Future Directions.

This article has been a first approach to cooperative game theory, and has emphasized two of its most important solution concepts. The literature on these topics is vast, and the interested reader is encouraged to consult the general references listed below. For the future, one should expect to see progress of the theory into areas that have been less explored, including games with asymmetric information and games with coalitional externalities. In both cases, the characteristic function model must be enriched to take care of the added complexities.

Relevant to this encyclopedia are issues of complexity. The complexity of cooperative solution concepts has been studied (see, for instance, [12]). In terms of computational complexity, the Shapley value seems to be easy to compute, while the core is harder, although some classes of games have been identified in which this task is also simple.

Finally, one should insist on the importance of novel and fruitful applications of the theory to shed new light on concrete problems. In the case of the core, for example, the insights of core stability in matching markets have been successfully applied by Alvin Roth and his collaborators to the design of matching markets in the "real world" (e.g., the job market for medical interns and hospitals, the allocation of organs from doners to patients, and so on) – see [31].

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