NASH PROGRAM

by

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Abstract: This article is a brief survey on the Nash program for coalitional games. Results of non-cooperative implementation of the Nash solution, the Shapley value and the core are discussed.

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Nash program

In game theory this is the name given to a research agenda, initiated in Nash (1953), intended to bridge the gap between the cooperative and non-cooperative approaches to the discipline.

Many authors have contributed to the program since its beginnings (the reader is referred to Serrano (2005) for a comprehensive survey). The current article will concentrate on a few salient contributions. One should begin by introducing some preliminaries and providing definitions of some basic concepts, extensions of which may be also found in other entries of the dictionary, such as those corresponding to non-cooperative games, coalitional games, the core, the Shapley value or bargaining.

1. Preliminaries. The non-cooperative approach to game theory provides a rich language and develops useful tools to analyze strategic situations. One clear advantage of the approach is that it is able to model how specific details of the interaction may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt.

Here are the primitives of the basic model in cooperative game theory. Let $N = \{1, \dots, n\}$ be a finite set of players. For each S, a non-empty subset of N, we shall specify a set $V(S)$ containing $|S|$-dimensional payoff vectors that are feasible for coalition S. Thus, a reduced form approach is taken because one does not explain what strategic choices are behind each of the payoff vectors in $V(S)$. In addition, in this formulation, referred to as the characteristic function, it is implicitly assumed that the actions taken by the complement coalition (those players not in S) cannot prevent S from achieving each of the payoff vectors in $V(S)$. There are more general models in which these sorts of externalities are considered, but for the most part the contributions to the Nash program have been confined to the characteristic function model. Given a collection of sets $V(S)$, one for each S, the theory formulates its predictions on the basis of solution concepts.

A solution is a mapping that assigns a set of payoff vectors in $V(N)$ to each characteristic function $(V(S))_{S\subseteq N}$. Thus, a solution in general prescribes a set, although it can be single-valued (when it assigns a unique payoff vector as a function of the fundamentals of the problem). The leading set-valued cooperative solution concept is the core, while the most used single-valued ones are the Nash bargaining solution and the Shapley value.

There are several criteria to evaluate the reasonableness or appeal of a cooperative solution. One could start by defending it on the basis of its definition alone. In the case of the core, this will be especially relevant: in a context in which players can freely get together in groups, the prediction should be payoff vectors that cannot be improved upon by any coalition. Alternatively, one can propose axioms, abstract principles, that one would like the solution to have, and the next step is to pursue their logical consequences. Historically, this was the first argument to justify the Nash solution and the Shapley value. However, some may think that the definition may be somewhat arbitrary, or one may object to the axiomatic approach for being "too abstract." By proposing non-cooperative games that specify the details of negotiation, the Nash program may help to counter these criticisms. First, the procedure will tell a story about how coalitions form and what sort of interaction among players is happening. In that process, because the tools of non-cooperative game theory are used for the analysis, the cooperative solution will be understood as the outcome of a series of strategic problems facing individual players. And second, novel connections and differences among solutions may now be uncovered from the distinct negotiation procedures that lead to each of them. Therefore, a result in the Nash program, referred to as a "non-cooperative foundation" or "non-cooperative implementation" of a cooperative solution, enhances its significance, being looked at now from a new perspective. Focusing on the features of the rules of negotiation that lead to different cooperative solutions takes one a long way in opening the "black box" of how a coalition came about, and contributes to a deeper understanding of the circumstances under which one solution versus another may be more appropriate to use.

2. The Nash bargaining solution. A particular case of a characteristic function is a two-player bargaining problem. In it, $N = \{1, 2\}$ is the set of players. The set $V(\{1, 2\})$, a compact and convex subset of $\mathbb{R}^2$, is the set of feasible payoffs if both players reach an agreement. Compactness may follow from the existence of a bounded physical pie that the parties are dividing, and convexity is a consequence of expected utility and the potential use of lotteries. The sets $(V(\{i\}))_{i \in N}$ are subsets of $\mathbb{R}$, and let $d_i = \max V(\{i\})$ be the disagreement payoff for player i, i.e., the payoff that i will receive if the parties fail to reach an agreement. It is assumed that $V(\{1, 2\})$ contains payoff vectors that Pareto dominate the disagreement payoffs. A solution assigns a feasible payoff pair to each bargaining problem.

This is the framework introduced in Nash (1950), where he proposes four axioms that a solution to bargaining problems should have. First, expected utility implies that, if payoff functions are rescaled via positive affine transformations, so must be the solution (scale invariance). Second, the solution must prescribe a Pareto efficient payoff pair (efficiency). Third, if the set $V(\{1, 2\})$ is symmetric with respect to the 45 degree line and $d_1 = d_2$, the solution must lie on that line (symmetry). And fourth, the solution must be independent of "irrelevant" alternatives, i.e., it must pick the same point if it is still feasible after one eliminates other points from the feasible set (IIA). Because of scale invariance, there is no loss of generality in normalizing the disagreement payoff to 0. We call the resulting problem a normalized problem.

Nash (1950) shows that there exists a unique solution satisfying scale invariance, efficiency, symmetry and IIA, and it is the one that assigns to each normalized bargaining problem the point (u_1, u_2) that maximizes the product $v_1 v_2$ over all $(v_1, v_2) \in V(\{1, 2\})$. Today we refer to this as the Nash solution. The use of the Nash solution is pervasive in applications, and following the axioms in Nash (1950), it is usually viewed as a normatively appealing resolution to bargaining problems.

In the first paper of the Nash program, Nash (1953) provides a non-cooperative approach to his axiomatically derived solution. This is done by means of a simple demand game. The two players are asked to demand simultaneously a payoff: player 1 demands v_1 and player 2 demands v_2. If the pair (v_1, v_2) is feasible, so that $(v_1, v_2) \in V(\{1, 2\})$, the corresponding agreement and split of the pie takes place to implement these payoffs. Otherwise, there is disagreement and payoffs are 0. To fix ideas, let us think of the existence of a physical pie of size 1 that is created if agreement is reached, while no pie is produced otherwise. Thus, player i's demand v_i corresponds to demanding a share x_i of the pie, $0 \leq x_i \leq 1$, such that player i's utility or payoff from receiving x_i is v_i.

The Nash demand game admits a continuum of Nash equilibria. Indeed, every point on the Pareto frontier of $V(\{1,2\})$ is a Nash equilibrium outcome, as is the disagreement payoff point if each player demands the payoff corresponding to having the entire pie. However, Nash (1953) introduces uncertainty concerning the exact size of the pie. Now players, when formulating their demands, must have to take into account that with some probability the pair of demands may lead to disagreement, even if they add up to less than 1. Then, it can be shown that the optimal choice of demands at a Nash equilibrium of the demand game with uncertain pie converges to the Nash solution payoffs as uncertainty becomes negligible. Hence, the Nash solution arises as the rule that equates marginal gain (through the increase in one's demanded share) and marginal loss (via the increase in the probability of disagreement) for each player when the problem is subject to a small degree of noise and demands/commitments are made simultaneously.

Rubinstein (1982) proposes a different non-cooperative procedure. In it, time preferences -impatience- and credibility of threats are the main forces that drive the equilibrium. The game is a potentially infinite sequence of alternating offers. In period 0, player 1 begins by making the first proposal. If player 2 accepts it, the game ends; otherwise, one period elapses and the rejector will make a counterproposal in period 1, and so on. Let $\delta \in [0,1)$ be the common per period discount factor, and let $v_i(\cdot)$ be player i's utility function over shares of the pie, assumed to be concave and strictly monotone. Thus, if player i receives a share x_i in an agreement reached in period t, his payoff is $\delta^{t-1}v_i(x_i)$. Perpetual disagreement has a payoff of 0.

Using subgame perfect equilibrium as the solution concept (the standard tool to rule out incredible threats in dynamic games of complete information), Rubinstein (1982) shows that there exists a unique prediction in his game. Specifically, the unique subgame perfect equilibrium prescribes an immediate agreement on the splits $(x, 1-x)$ –offered by player 1– and $(y, 1-y)$ –by player 2–, which are described by the following equations:

$$v_1(y) = \delta v_1(x)$$
$$v_2(1-x) = \delta v_2(1-y).$$

That is, at the unique equilibrium, the player acting as a responder in a period is offered a share that makes him exactly indifferent between accepting and rejecting it to play the continuation: the bulk of the proof is to show that any other behavior relies on incredible threats.

As demonstrated in Binmore, Rubinstein and Wolinsky (1986), the unique equilibrium payoffs of the Rubinstein game, regardless of who is the first proposer, converge to the Nash solution payoffs as $\delta \to 1$. First, note that the above equations imply that, for any value of δ, the product of payoffs $v_1(x)v_2(1-x)$ is the same as the product $v_1(y)v_2(1-y)$. Thus, both points, $(v_1(x), v_2(1-x))$ and $(v_1(y), v_2(1-y))$, lie on the same hyperbola of equation $v_1 v_2 = K$, and in addition, since they correspond to efficient agreements, both points also lie on the Pareto frontier of $V(\{1,2\})$. Finally, as $\delta \to 1$, one has that $x \to y$ so that the two proposals (the one made by player 1 and the other by player 2) converge to one and the same, the one that yields the Nash solution payoffs. Thus, credible threats in dynamic negotiations in which both players are equally and almost completely patient also lead to the Nash solution.

3. The Shapley value. Now consider an n-player coalitional game where payoffs are transferable in a one-to-one rate among different players (for instance, because utility is money for all of them). This means that $V(S)$, the feasible set for coalition S, is the set of payoffs $(x_i)_{i\in S}$ satisfying the inequality $\sum_{i\in S} x_i \leq v(S)$ for some real number $v(S)$. This is called a transferable utility or TU game in characteristic function form. The number $v(S)$ is referred to as the worth of S, and it expresses S's initial position (e.g., the maximum total utility that the group S of agents can achieve in an exchange economy by redistributing their endowments when utility is quasilinear).

Therefore, without loss of generality, we can describe a TU game as a collection of real numbers $(v(S))_{S\subseteq N}$. A solution is then a mapping that assigns to each TU game a set of payoffs in the set $V(N)$, i.e., vectors $(x_1, \ldots, x_n)$ such that $\sum_{i\in N} x_i \leq v(N)$. In this section, as in the last one, we shall require that the solution be single-valued.

Shapley (1953) is interested in solving in a fair way the problem of distribution of surplus among the players, when taking into account the worth of each coalition. To do this, he resorts to the axiomatic method. First, the payoffs must add up to $v(N)$, which means that all the surplus is allocated (efficiency). Second, if two players are substitutes because they contribute the same to each coalition, the solution should treat them equally (symmetry). Third, the solution to the sum of two TU games must be the sum of what it awards to each of the two games (additivity). And fourth, if a player contributes nothing to every coalition, the solution should pay him nothing (dummy).

The result in Shapley (1953) is that there is a unique single-valued solution to TU games satisfying efficiency, symmetry, additivity and dummy. It is what today we call the Shapley value, the function that assigns to each player i the payoff

$$\mathrm{Sh}_i(N, v) = \sum_{S, i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}[v(S) - v(S \setminus \{i\})].$$

That is, the Shapley value awards to each player the average of his marginal contributions to each coalition. In taking this average, all orders of the players are considered to be equally likely. Let us assume, also without loss of generality, that $v(\{i\}) = 0$ for each player i.

Hart and Mas-Colell (1996) propose the following non-cooperative procedure. With equal probability, each player $i \in N$ is chosen to publicly make a feasible proposal to the others: $(x_1, \ldots, x_n)$ is such that the sum of its components cannot exceed $v(N)$. The other players get to respond to it in sequence, following a prespecified order. If all accept, the proposal is implemented; otherwise, a random device is triggered. With probability $0 \leq \delta < 1$, the same game continues being played among the same n players (and thus, a new proposer will be chosen again at random among them), but with probability $1 - \delta$, the proposer leaves the game. He is paid 0 and his resources are removed, so that in the next period, proposals to the remaining $n - 1$ players cannot add up to more than $v(N \setminus \{i\})$. A new proposer is chosen at random among the set $N \setminus \{i\}$, and so on.

As shown in Hart and Mas-Colell (1996), there exists a unique stationary subgame perfect equilibrium payoff profile of this procedure, and it actually coincides with the Shapley value payoffs for any value of δ. (Stationarity means that strategies cannot be history dependent). As $\delta \to 1$, the Shapley value payoffs are also obtained not only in expectation, but with independence of who is the proposer. One way to understand this result, as done in Hart and Mas-Colell (1996), is to check that the rules of the procedure and stationary behavior in it are in agreement with Shapley's axioms. That is, the equilibrium relies on immediate acceptances of proposals, stationary strategies treat substitute players similarly, the equations describing the equilibrium have an additive structure, and dummy players will have to receive 0 because no resources are destroyed if they are asked to leave. It is also worth stressing the important role in the procedure of players' marginal contributions to coalitions: following a rejection, a proposer incurs the risk of being thrown out and the others of losing his resources, which seem to suggest a "price" for them.

4. The core. The idea of agreements that are immune to coalitional deviations was first introduced to economic theory in Edgeworth (1881), who defined the set of coalitionally stable allocations of an economy under the name "final settlements." Edgeworth envisioned this concept as an alternative to Walrasian equilibrium (Walras (1874)), and was also the first to investigate the connections between the two concepts. Edgeworth's notion, which today we refer to as the core, was rediscovered and introduced to game theory in Gillies (1959). Therefore, the origins of the core were not axiomatic. Rather, its simple definition appropriately describes stable outcomes in a context of unfettered coalitional interaction. (The axiomatizations of the core came much later: e.g., Peleg (1985, 1986), Serrano and Volij (1998).)

For simplicity, let us continue to assume that we are studying a TU game. In this context, the core is the set of payoff vectors $x = (x_1, \ldots, x_n)$ that are feasible, i.e., $\sum_{i \in N} x_i \leq v(N)$, and such that there does not exist any coalition $S \subseteq N$ for which $\sum_{i \in S} x_i < v(S)$. If such a coalition S exists, we shall say that S can improve upon or block x, and x is deemed unstable. The core usually prescribes a set of payoffs, instead of a single one, and also, it can prescribe the empty set in some games.

To obtain a non-cooperative implementation of the core, the procedure must embody some feature of anonymity, since the core is usually a large set and it contains payoffs where different players are treated very differently. Perry and Reny (1994) build in this anonymity by assuming that negotiations take place in continuous time, so that anyone can speak at the beginning of the game, instead of having a fixed order. The player that gets to speak first makes a proposal consisting of naming a coalition that contains him and a feasible payoff for that coalition. Next, the players in that coalition get to respond. If they all accept the proposal, the coalition leaves and the game continues among the other players. Otherwise, a new proposal may come from any player in N. It is shown that, if the TU game has a non-empty core (as well as any of its subgames), the stationary subgame perfect equilibrium outcomes of this procedure coincide with the core. If a core payoff is proposed to the grand coalition, there are no incentives for individual players to reject it. Conversely, a non-core payoff cannot be sustained because any player in a blocking coalition has an incentive to make a proposal to that coalition, who will accept it (knowing that the alternative, given stationarity, would be to go back to the non-core status quo). Moldovanu and Winter (1995) offer a discrete-time version of the mechanism: in their work, the anonymity required is imposed on the solution concept, by looking at order-independent equilibria.

Serrano (1995) sets up a market to implement the core. The anonymity of the procedure stems from the random choice of broker. The broker announces a vector $(x_1, \ldots, x_n)$, where the components add up to $v(N)$. One can interpret x_i as the price for the productive asset held by player i. Following an arbitrary order, the remaining players either accept or reject these prices. If player i accepts, he sells his asset to the broker for the price x_i and leaves the game. Those who reject get to buy from the broker, at the called out prices, the portfolio of assets of their choice if the broker still has them. If a player rejects, but does not get to buy the portfolio of assets he would like because someone else took them before, he can always leave the market with his own asset. The broker's payoff is the worth of the final portfolio of assets that he holds, plus the net monetary transfers that he has received. Serrano (1995) shows that the prices announced by the broker will always be his top-ranked vectors in the core. If the TU game is such that gains from cooperation increase with the size of coalitions, the set of all subgame perfect equilibrium payoffs of this procedure will coincide with the core. Core payoffs are here understood as those price vectors where all arbitrage opportunities in the market have been wiped out. Finally, yet another way to build anonymity in the procedure is by allowing the proposal to be made by brokers outside of the set N, as done in Pérez-Castrillo (1994).

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