Lecture Notes: September 14, 2005

1. The Production Function.

$$Y = F(K, L) = AK^{\alpha}L^{\beta},$$

where Y is the level of Output (or equivalently, Income); A is a constant Productivity multiple; K is the level of Capital; and L is the number of Workers in the economy. For now, consider α and β as simply unchanging parameters.

A function, written in this manner (multiplicative with exponents) is usually referred to as a Cobb-Douglas Production Function. Any C-D function can display one of three properties:

1) Constant Returns to Scale (CRS)
2) Increasing Returns to Scale (IRS)
3) Decreasing Returns to Scale (DRS)

The property the C-D function displays depends on the parameters α and β. For our given production function only, if…

$\alpha + \beta = 1$ then CRS
$\alpha + \beta > 1$ then IRS
$\alpha + \beta < 1$ then DRS

Why: If the each quantity is multiplied by a factor z,

$$F(zK, zL)$$
$$A(zK)^{\alpha}(zL)^{\beta}$$
$$A(K)^{\alpha}(L)^{\beta}z^{\alpha+\beta}$$
$$z^{\alpha+\beta} * F(K, L)$$
$$z^{\alpha+\beta} * Y$$

So, the summed value of the exponents determines the returns to scale.

For the remainder of this class, we will ***assume*** CRS C-D production functions.

2. Implications of a CRS Cobb-Douglas Production function.

Recall the concept of marginal products and the concept of diminishing at the margin. When this production function is CRS, there will be diminishing marginal product in capital and labor. In other words, for a given amount of workers, the return on output from adding one additional unit of capital is the marginal product. As capital continues to be added, the return from each additional unit declines (the concept of the diminishing marginal product of capital).

Mathematically,

$$MPK = \frac{\partial F(K,L)}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha},$$
$$\frac{\partial MPK}{\partial K} = \frac{\partial^2 F(K,L)}{\partial K^2} = \frac{\partial \alpha A K^{\alpha-1} L^{1-\alpha}}{\partial K} = \alpha(\alpha-1) A K^{\alpha-2} L^{1-\alpha} < 0.$$

3. In per-worker terms.

Since we are not necessarily interested in the amount of capital in a country, but rather in the amount of capital available for each worker, we can algebraically manipulate the production function into per-worker terms. Dividing both sides by L and then writing the analogous per-worker version as lower case variables,

$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L},$$
$$\frac{Y}{L} = \frac{AK^{\alpha}L^{1-\alpha}}{L^{\alpha}L^{1-\alpha}} = A\left(\frac{K}{L}\right)^{\alpha}\left(\frac{L}{L}\right)^{1-\alpha} = A\left(\frac{K}{L}\right)^{\alpha},$$
$$y = Ak^{\alpha}.$$

It is important to note that the marginal product of capital per worker also diminishes in regards to output per worker. (Can you show this?)

Given the fact that (1) the slope of the production function (the first derivative and equivalently, the marginal product) is positive and (2) the second derivative is negative meaning that the positive slope 'becomes less positive' (diminishing marginal product), we have the tools to graph the production function with k on the horizontal axis and y on the vertical axis. (Basically, it is a line, bowed-out.)

4. Growth Rates – Continuous.

Δk is the change in k, and $\frac{\Delta k}{k}$ is the growth rate of k over that year. However, for continuous growth rates, we use the following formulation, $\dot{k}$ as the change in k over time and similarly, $\frac{\dot{k}}{k}$ as the growth rate of k over time. Mathematically, we can covert an equation into growth rates because of the unique properties in the derivative of the natural log.

$$\frac{\partial \ln(x)}{\partial x} = \frac{\left(\frac{d}{dx}(x)\right)}{x} = \frac{1}{x}.$$

Instead of taking the derivative a variable by *x* and instead, we take the derivative with respect to time (how much does *x* change when time changes), we get:

$$\frac{\partial \ln(x)}{\partial t} = \frac{\left(\frac{\partial x}{\partial t}\right)}{x} = \frac{\dot{x}}{x}$$, the growth rate of *x*.

For our production function, by taking the natural log of both sides and taking the derivative with respect to time,

$$y = Ak^{\alpha}$$
$$\ln(y) = \ln(Ak^{\alpha}) = \ln(A) + \ln(k^{\alpha}) = \ln(A) + \alpha \ln(k)$$
$$\frac{\partial \ln(y)}{\partial t} = \frac{\partial[\ln(A) + \alpha \ln(k)]}{\partial t}$$
$$\frac{\partial \ln(y)}{\partial t} = \frac{\partial[\ln(A)]}{\partial t} + \frac{\partial[\alpha \ln(k)]}{\partial t}$$
$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \alpha \frac{\dot{k}}{k}$$
$$\hat{y} = \hat{A} + \alpha \hat{k}$$

The result is that the growth rate of output is proportional, by a factor α, to the growth rate of capital with our given assumptions!