Lecture Notes: September 19, 2005

1. What is α ?

Recall that factors are paid their marginal products. To find the marginal product of capital, we need to use the non-per-worker modified equation. We have already done this. Now, because each factor is paid their marginal product, and we have *x* number of that particular factor, we multiply the marginal product by the amount to find the total amount paid to that factor. This total amount, as a percentage of total output is the exponent on that factor!

$$\frac{MPK * K}{Y} = \frac{\left(\alpha AK^{\alpha-1}L^{1-\alpha}\right)(K)}{AK^{\alpha}L^{1-\alpha}} = \frac{\alpha\left(AK^{\alpha}L^{1-\alpha}\right)}{AK^{\alpha}L^{1-\alpha}} = \alpha.$$

$$\frac{MPL * L}{Y} = \frac{\left((1-\alpha)AK^{\alpha}L^{1-\alpha-1}\right)(L)}{AK^{\alpha}L^{1-\alpha}} = \frac{(1-\alpha)\left(AK^{\alpha}L^{1-\alpha}\right)}{AK^{\alpha}L^{1-\alpha}} = 1-\alpha.$$

In fact, this property is not by coincidence. The Cobb-Douglas production function is named after Cobb and Douglas for the reason that they wanted a function that had this property. Empirical research shows that α is approximately and likely one-third.

2. How *k* changes.

k is the amount of capital per worker in a given country. Because we are attempting to isolate the effect of capital, only, we will not concern ourselves now with changes in the number of workers. So, this begs the question: What makes capital grow and what makes capital shrink? Capital is produced through investment in capital. Capital is lost through depreciation. We will concern ourselves with the rates of investment and the rates of depreciation to determine the total amount of investment and depreciation.

There are two principal determinants of capital intensity:

1) the investment effort: the effort of savings and investment that increases the amount of new capital that goes to equipping workers for increasing output
2) the investment requirements: the amount of the investment that goes to simply replacing the normal wear-and-tear of machines.

Now, let's formalize this concept:

Start with an amount of capital/worker (k_{today}) today, where the subscript indicates the appropriate time. We are going to account for two factors: (1) how much capital we are going to add (the investment effort that we make) and (2) how much capital we are going to lose (what are the investment requirements)

k_{today} + *investment – depreciation*

$k_{today} + \gamma(y) - \delta k_{today}$

The above expression just says that from the amount of capital per worker we have today, we will produce a certain amount of output per worker and a certain amount of that (γ) will be invested into making more capital per worker whereas, we will lose a fraction δ of the existing amount of *k*. When these effects are accounted for, we will have the amount of capital per worker tomorrow.

$k_{tomorrow} = k_{today} + \gamma(y) - \delta k_{today}$

If we rearrange the equation,

$k_{tomorrow} - k_{today} = \gamma(y) - \delta k_{today}$

$\Delta k = \gamma(y) - \delta k_{today}$

$$\textit{Change in Capital} = \Delta k = \gamma y - \delta k = \gamma\left(Ak^{\alpha}\right) - \delta k.$$

The above expression tells us how much capital per worker changes from year to year. We now graph this equation. γ is the investment rate and δ is the depreciation rate. If these are the rates, then the *amounts* will be, γy where γ percent of output is the total investment and δk where δ percent of the existing capital stock will be depreciated.

(graph)

What I want you to take away from this is how *k* changes over time.

To find the growth rate of capital over time, divide the above equation by *k*.

$$\frac{\Delta k}{k} = \frac{\gamma\left(Ak^{\alpha}\right) - \delta k}{k} = \gamma A k^{\alpha-1} - \delta.$$

3. Stable Steady States.

For the remaining analysis in this section, we isolate the role of capital by keeping constant, the depreciation rate, as well as productivity. Anyhow, why do we care about a steady state? We care because the steady state is a predicted state of the world. Why is it predicted? It is a direct result of the equation showing how capital changes over time.

Capital can be growing, shrinking, or not changing. As a thought exercise, for a given investment rate and depreciation rate, think about a country in which capital is growing. This implies that the absolute amount invested is greater than the absolute amount depreciating. The rates remain constant, so, with increased capital accumulation, you will get to a point where the absolute amount that depreciates is greater than the absolute amount you gain. Basically, if capital kept on growing, it would have to shrink. In actuality, the constant rates of investment and depreciation mean that capital will simply stop growing altogether and vice versa if you start from shrinking capital. The point at which this happens is called the steady state. Naturally, it is characterized by the fact that $\Delta k = 0$.

To find the steady state, we use that condition:

$$\Delta k = 0$$

$$0 = \gamma(Ak^{\alpha}) - \delta k$$

$$\gamma(Ak^{\alpha}) = \delta k$$

$$\frac{\gamma A}{\delta} = \frac{k}{k^{\alpha}}$$

$$\frac{\gamma A}{\delta} = k^{1-\alpha}$$

$$\left(\frac{\gamma A}{\delta}\right)^{\left(\frac{1}{1-\alpha}\right)} = k_{ss}$$

and

$$y_{ss} = Ak_{xx}^{\alpha} = A\left(\left(\frac{\gamma A}{\delta}\right)^{\left(\frac{1}{1-\alpha}\right)}\right)^{\alpha} = A\left(\left(\frac{\gamma A}{\delta}\right)^{\left(\frac{\alpha}{1-\alpha}\right)}\right). = A^{\left(1+\frac{\alpha}{1-\alpha}\right)}\left(\frac{\gamma}{\delta}\right)^{\left(\frac{\alpha}{1-\alpha}\right)} = A^{\left(\frac{1}{1-\alpha}\right)}\left(\frac{\gamma}{\delta}\right)^{\left(\frac{\alpha}{1-\alpha}\right)}.$$