Problem Set 2.
Solutions

I. To find the steady-state value of the country, we use the following formulation derived from setting the motion of capital equation to zero, solving for k and then solving for y. Ultimately, we get

$$y_{ss} = A^{\frac{1}{1-\alpha}}\left(\frac{\gamma}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Plugging in values: $A = 1$, $\alpha = 0.5$, $\gamma = 0.5$, and $\delta = 0.05$, we get:

$$y_{ss} = 1^{\frac{1}{1-0.5}}\left(\frac{0.5}{0.05}\right)^{\frac{0.5}{1-0.5}}.$$

Simplifying the above equation, we get $y_{ss} = 10$.

To find the current output per worker, we substitute in k=400 into the production function to get:

$$y = k^{\frac{1}{2}} = 400^{\frac{1}{2}} = 20.$$

That is, the current output is 20 whereas the steady-state output level is 10. Therefore, we conclude that $y > y_{ss}$ and so the country is above its steady-state level of output per worker.

II.

(a) First we find the steady-state level of capital per worker. Using the values for investment, γ =0.25 , depreciation, δ = 0.05, productivity, A=1, and α =0.5, we get,

$$k_{ss} = \left(\frac{A\gamma}{\delta}\right)^{\frac{1}{1-\alpha}} = \left(\frac{(1)(0.25)}{0.05}\right)^{\frac{1}{1-0.5}} = 5^2 = 25.$$

That is, the steady-state level of capital per worker is 25. Plugging in k_{ss} into the production function we get the steady-state level of output per worker to be:

$$y_{ss} = k_{ss}^{\frac{1}{2}} = (25)^{\frac{1}{2}} = 5.$$

That is, the steady-state level of output per worker is 5.

(b) For year 2, using 16.2 as the value for capital per worker, calculate output, y, followed by investment γy, depreciation δk, and then change in capital stock. Add the value for change in capital stock to 16.2, the value for capital per worker in year 2, to get capital per worker for year 3. Use year 3 capital to obtain all the values for year 3 and continue up to year 8. The filled in table is below.

Year	Capital	Output	Investment	Depreciation	Change in Capital Stock
1	16.00	4.00	1.00	0.80	0.20
2	16.20	4.02	1.01	0.81	0.20
3	16.40	4.05	1.01	0.82	0.19
4	16.59	4.07	1.02	0.83	0.19
5	16.78	4.10	1.02	0.84	0.19
6	16.96	4.12	1.03	0.85	0.18
7	17.14	4.14	1.04	0.86	0.18
8	17.32	4.16	1.04	0.87	0.17

(c) The growth rate of output between years 1 and 2 is given by:

$$g = \left(\frac{y_2}{y_1}\right) - 1 = \left(\frac{4.02}{4}\right) - 1 = 0.005.$$

That is, output per worker grew at a rate of 0.5 % between years 1 and 2. (Using exact values, the growth rate is approximately 0.62 % for years 1 and 2.)

(d) The growth rate of output between years 7 and 8 is given by:

$$g = \left(\frac{y_8}{y_7}\right) - 1 = \left(\frac{4.16}{4.14}\right) - 1 = 0.0048.$$

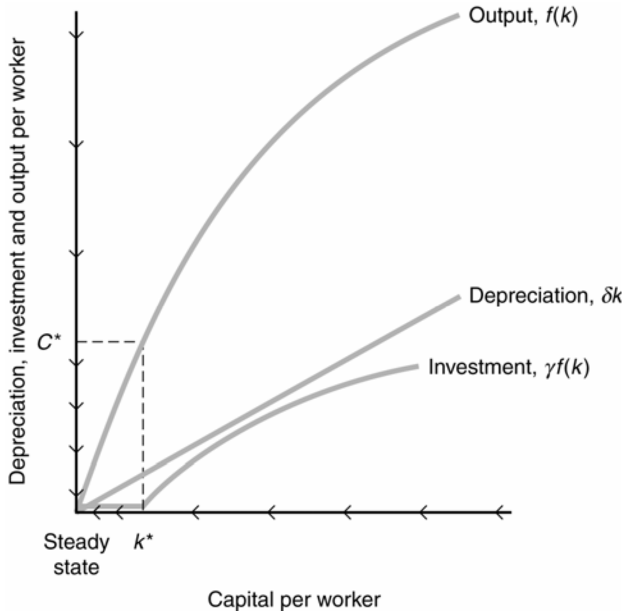
That is, output per worker grew at a rate of 0.48 % between years 7 and 8. (Using exact values, the growth rate is approximately 0.52 % for years 7 and 8.)

(e) The speed of growth has changed from 0.50 % to 0.48 % implying that growth has slowed down at a rate of 4 %. Thus, as a country reaches their steady-state value, the rate of growth slows.

III. Before beginning the analysis, we define two new variables. The level of capital per worker, necessary to achieve consumption level c^* is denoted k^*. Technically, k^* is given by $f^{-1}(c^*)$. Therefore, if the initial level of capital, k_i is above k^*, savings will be positive, and if k_i is below k^*, savings will 0. The second variable I define is $\bar{k}$ (refer to second figure). It is the level of capital at which depreciation is equal to savings and distinct from the steady-state level of capital, if it exists. We are now ready to begin our analysis. There are two cases.

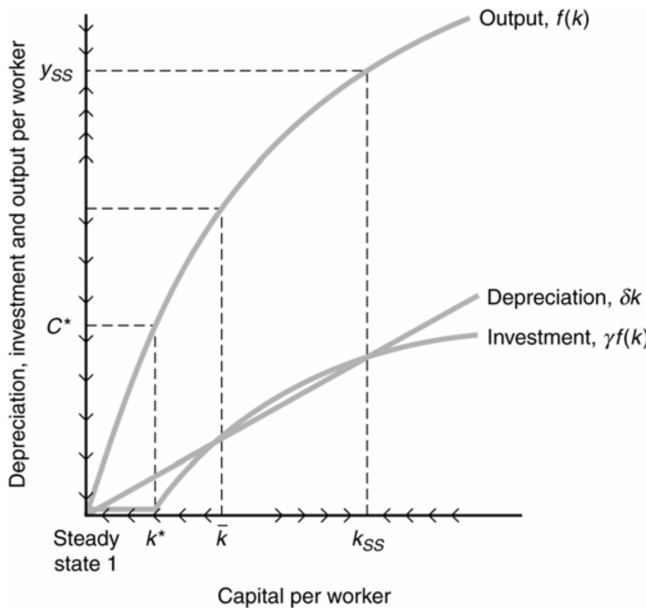
<u>Case 1.</u> Depreciation is always greater than Saving; $\delta k > \gamma f(k), \forall k$

In the figure below, at any initial level of k_i, depreciation is always greater than savings. The level of capital falls over time, as does the level of income per worker. Consequently, the economy will continue to stagnate until the level of income and the capital stock are zero.



Case 2. Depreciation is not always greater than Saving; $\delta k \leq \gamma f(k), \textit{for some } k$

The figure below shows two possible scenarios. If the initial level of capital, k_i, is equal to or below k^*, then savings in the economy will be zero. The level of capital in the economy falls due to depreciation and we achieve the same result as in the first case. On the other hand, if $k_i \geq \bar{k}$, then the level of savings exceeds or will exceed the level of depreciation and the capital stock rises over time. The capital stock will reach a state-state value as will income. If $k^* < k_i < \bar{k}$, then the amount of savings does not exceed the amount of depreciation. The level of capital stock begins to fall and we are in the first case where both income and capital go to zero levels. In the end, the ultimate determinant of where the economy rests is determined by the initial level of capital, commensurate with the initial level of income.



Case III. (just to be precise). There is one last possibility of a steady state and this would occur when the depreciation curve is tangent to the investment curve. In this scenario, the point of tangency is the steady state. Note that this situation is an extreme version of Case II.

IV. For there to exist single steady states, two conditions must hold. First, there must be only one level of capital per worker such that capital dilution and capital depreciation equal investment in capital. Pictorially, there should be only one intersection. Second, the economy must converge to this point from any initial starting level of capital. To ensure that both criteria hold, we have two possible cases. The first case is such that the level of investment exceeds the high capital dilution and depreciation regime, for any given level of capital that dictates a high rate of population growth. In the crappy 'word art' figure below, it is the higher savings curve. The steady state in this case is the level of capital commensurate with the intersection between the low capital dilution and depreciation curve and the investment curve. The second case of a single steady state is given by a rate of savings such that it intersects with only the high capital dilution and depreciation curve and not with the lower regime. This is possible because at high levels of capital per worker, investment need not necessarily exceed the lower level of capital dilution and depreciation due to diminishing returns to capital. The figure is not drawn, but try to draw a savings curve whose curvature remains positive but gets 'flatter' over time. This curve is bound to intersect with the high population growth regime, but it can be shown to not intersect with the low population growth regime. The mathematical conditions for the savings rates are given below.

Single: $\gamma > (n_H + \delta)\bar{k}^{1-\alpha}$ (as the figure shows)

In this case, we end up with a positive and unique steady state in the low population growth regime.

$\gamma < (n_L + \delta)\bar{k}^{1-\alpha}$ (not drawn in the figure below)

In this case, we end up with a positive and unique steady state in the high population growth regime.

For the case of multiple steady states, we need investment curves to cross both high population and low population growth regimes. The conditions on the savings rates are provided below.

Multiple: $(n_H + \delta)\bar{k}^{1-\alpha} > \gamma > (n_L + \delta)\bar{k}^{1-\alpha}$ (as drawn in figure below)

This follows from the fact that there will be two steady state levels of capital stock per worker, and depending on your initial value of *k*, you will converge and stay at one or the other.