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A typical allocation in an N-replica economy is:

$$x = (x_{hm}) = (x_{11}, \ldots, x_{1N}, \ldots x_{H1}, \ldots x_{HN}) \in R_{+}^{LHN}.$$

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Proposition 18.B.2: Suppose x^* belongs to the core of the N-replica economy. Then x^* satisfies the equal-treatment property, i.e., consumers of the same type consume the same bundle:

$$x^*_{hm} = x^*_{hn} \text{ for all } 1 \leq m, n \leq N \text{ and all } 1 \leq h \leq H.$$

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Proof: Suppose $x \in R^{LHN}$ does not have the equal treatment property. Wlog, this means that $x_{1m} \neq x_{1n}$ for some m and n.

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Proof: Suppose $x \in R^{LHN}$ does not have the equal treatment property. Wlog, this means that $x_{1m} \neq x_{1n}$ for some m and n.

We will show that x is not in the core - by showing that x can be improved upon by a coalition of H consumers, formed by choosing from every type a worst-treated individual among the consumers of that type.

Wlog, suppose that for every h, $h1$ is one such worst-treated individual, i.e.,

$$x_{hn} \succeq_h x_{h1} \text{ for all } h \text{ and } n.$$

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By strict convexity:

$$\hat{x}_h \succeq_h x_{h1} \text{ for all } h \text{ and } \hat{x}_1 \succ_1 x_{11}. \qquad (18.B.1)$$

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We claim that coalition $S = \{11, \ldots, h1, \ldots H1\}$, formed by H consumers, can get, with its own endowments, the commodity bundles $(\hat{x}_1, \ldots, \hat{x}_H)$. By (18.B.1) this would mean that S can improve upon x.

It remains to be checked that $\widehat{x}$ is feasible.

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$$\sum_h \hat{x}_h = \frac{1}{N} \sum_h (\sum_n x_{hn}) \leq \frac{1}{N}[N \sum_h \omega_h] = \sum_h \omega_h,$$

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This result allows us to regard the core allocations as vectors of fixed size LH, irrespective of the replica that we are concerned with. As a matter of terminology, we call a vector $(x_1, \ldots, x_H) \in R_+^{LH}$ a type allocation and, for any replica N, interpret it as the equal-treatment allocations where each consumer of type h gets x_h.

Note that the core allocations of an N replica economy are in R^{LNH}. Strictly speaking, this means that we cannot compare the core allocations of an N replica with those of an N' replica.

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Let $C_N \subset R_+^{LH}$ denote the set of type allocations corresponding to the (equal treatment) core allocations of the N-replica economy.

Let $W_N \subset R_+^{LH}$ denote the type allocations corresponding to the (equal treatment) Walrasian allocations of the N-replica economy.

Observation 1. Suppose coalition S has an objection against a type allocation x in the N-replica. Then the same is true in the $N+1$-replica. This implies:

$$C_{N+1} \subseteq C_N \text{ for all } N.$$

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Observation 2. If a type allocation x is Walrasian in the N-replica it is also Walrasian in the $N+1$-replica, and vice versa:

$$W_N = W_{N+1} = W \text{ for all } N.$$

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Putting together these observations:

$$W \subseteq \ldots \subseteq C_{N+1} \subseteq C_N \subseteq \ldots \subseteq C_1.$$

Theorem (Debreu-Scarf):

$$\cap_N C_N = W,$$

i.e., if $x^* \in C_N$ for all N, then x^* is Walrasian.