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Thus the basic economy is $(\{(X_i, \succeq_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega})$.

A pure exchange economy is one in which $Y_j = -\Re^L_+$ for all j.

An allocation $(x, y) = ((x_i)_{i=1}^{I}, (y_j)_{j=1}^{J})$ consists of commodity bundles $x_i \in X_i$ for all i and production plans $y_j \in Y_j$ for each j.

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The set of feasible allocations, a subset of $\Re^{L(I+J)}$, is:

$$A = \{(x, y) \in \prod_i X_i \times \prod_j Y_j | \sum_i x_i = \sum_j y_j + \bar{\omega}\}.$$

A feasible allocation, (x, y), is Pareto optimal if there does not exist another feasible allocation (x', y') such that

$$x'_i \succeq_i x_i \text{ for all } i \text{ and } x'_i \succ_i x_i \text{ for some } i.$$

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A feasible allocation, (x, y), is weakly Pareto optimal if there does not exist another feasible allocation (x', y') such that

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Private Ownership Economies

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Each consumer i has an endowment $\omega_i \in \Re^L$ and, in each firm j, a share θ_{ij} where $0 \leq \theta_{ij} \leq 1$. Naturally, we require that:

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A private ownership economyis therefore defined as:

$$\mathcal{E} = (\{X_i, \succeq_i, \omega_i, (\theta_{ij})_{j=1}^J\}_{i=1}^I, \{Y_j\}_{j=1}^J).$$

Definition 16.B.3. Given a private ownership economy specified by $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1}, ..., \theta_{iJ})\}_{i=1}^{I})$, an allocation (x^*, y^*) and a price vector $p = (p_1, ..., p_L)$ constitute a **Walrasian (or competitive) equilibrium** if:

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(i) For every j, $\quad p \cdot y_j \leq p \cdot y_j^* \quad$ for all $y_j \in Y_j$.

(ii) For every i, $\quad p \cdot x_i^* \leq w_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$ and
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Definition 16.B.4. Given an economy specified by $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega}\}$, an allocation (x^*, y^*) and a price vector $p = (p_1, ..., p_L)$ constitute a **price equilibrium with transfers** if there is an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

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(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Proposition 16.C.1. (First Fundamental Theorem of Welfare Economics). If preferences are locally nonsatiated, and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

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Proof: Suppose that (x^*, y^*, p) is a price equilibrium with transfers and that the associated wealth levels are $(w_1, ..., w_I)$. Recall that $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

The preference maximization part of the definition of a price equilibrium with transfers [i.e., part (ii) of Definition 16.B.4] implies that

$$\text{If } x_i \succ_i x_i^* \qquad \text{then } p \cdot x_i > w_i. \tag{16.C.1}$$

By local non-satiation, we have the additional property:

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Now consider an allocation (x, y) that Pareto dominates (x^*, y^*). That is, $x_i \succeq_i x_i^*$ for all i and $x_i \succ_i x_i^*$ for some i.

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By (16.C.2), we must have $p \cdot x_i \geq w_i$ for all i, and by (16.C.1) $p \cdot x_i > w_i$ for some i. Hence,

$$\sum_i p \cdot x_i > \sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*.$$

Moreover, because y_j^* is profit maximizing for firm j at price vector p, we have $p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$. Thus,

Moreover, because y_j^* is profit maximizing for firm j at price vector p, we have $p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$. Thus,

$$\sum_i p \cdot x_i > p \cdot \bar{\omega} + \sum_j p \cdot y_j. \tag{16.C.3}$$

Moreover, because y_j^* is profit maximizing for firm j at price vector p, we have $p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$. Thus,

$$\sum_i p \cdot x_i > p \cdot \bar{\omega} + \sum_j p \cdot y_j. \qquad (16.C.3)$$

But then (x, y) cannot be feasible. Indeed, $\sum_i x_i = \bar{\omega} + \sum_j y_j$ implies $\sum_i p \cdot x_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j$, which contradicts (16.C.3). We conclude that the equilibrium allocation (x^*, y^*) must be Pareto optimal. ■

Definition 16.D.1. Given an economy specified by $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega}\}$, an allocation (x^*, y^*) and a price vector $p = (p_1, ..., p_L) \neq 0$ constitute a **price quasi-equilibrium with transfers** if there is an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

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Consider an economy specified by $(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega})$, and suppose that every Y_j is convex and every preference relation $\succeq_i$ is convex [i.e., the set $\{x_i' \in X_i : x_i' \succeq_i x_i\}$ is convex for every $x_i \in X_i$] and locally non-satiated.

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Then, for every Pareto optimal allocation (x^*, y^*), there exists a price vector $p = \{p_1, \ldots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasi-equilibrium with transfers.

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By the convexity assumptions these two disjoint sets are both convex.

By the separating hyperplane theorem, there exists $p \neq 0$ and a number r such that

$$p \cdot z \geq r \text{ for all } z \in V \text{ and } p \cdot z \leq r \text{ for all } z \in Y + \{\bar{\omega}\}.$$

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Claim: $r = p \cdot x^* = p \cdot y^* + p \cdot \bar{\omega}$.

Proof: Since $p \cdot x \geq r$ for all $x \succ x^*$, it follows from LNS that $p \cdot x^* \geq r$.

But $x^* = y^* + \bar{\omega}$ and $p \cdot y + p \cdot \bar{\omega} \leq r$ now yields the conclusion. ■

Thus we have found $p \neq 0$ such that:

- If $x' \succ x^*$ then $p \cdot x' \geq p \cdot x^* = p \cdot (y^* + \bar{\omega})$

- $p \cdot y^* \geq p \cdot y$ for all $y \in Y$

- $x^* = y^* + \bar{\omega}$ (given by hypothesis)

i.e., (x^*, y^*, p) is a price quasi equilibrium (with transfers).

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If preferences satisfy LNS, then (B) implies (C).

If preferences are continuous and there exist locally cheaper points (e.g. if $w_i > 0$ and $X_i = \Re_+^L$), then (B) implies (A); see Proposition 16.D.2.

Proof of Proposition 16.D.1

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Given the Pareto optimal allocation (x^*, y^*), for each i, let $V_i = \{x_i \in X_i : x_i \succ_i x_i^*\} \subset \Re^L$ and define

$$V = \sum_i V_i = \{ \sum_i x_i \in \Re^L : x_1 \in V_1, ..., x_I \in V_I\}.$$

Let Y denote the aggregate production set:

$$Y = \sum_j Y_j = \{ \sum_i y_i \in \Re^L : y_1 \in Y_1, ..., y_J \in Y_J\}.$$

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Step 1: Every set V_i is convex. Suppose that $x_i \succ_i x_i^*$ and $x_i' \succ_i x_i^*$. Take $0 \leq \alpha \leq 1$. We want to prove that $\alpha x_1 + (1-\alpha)x_i' \succ_i x_i^*$. (Note the typo in the book). Because preferences are complete, we can assume w.l.o.g. $x_i \succeq_i x_i'$. Therefore, by convexity of preferences, we have $\alpha x_i + (1-\alpha)x_i' \succeq_i x_i'$, which by transitivity yields the desired conclusion: $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$.

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This follows from Pareto optimality. Suppose there was $z \in V \cap (Y + \{\bar{\omega}\})$. This means that there exists $x_i \in V_i$ and $y_j \in Y_j$ such that

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i.e., (x, y) is a Pareto improvement over (x^*, y^*), a contradiction.

Step 4: There is $p = (p_1, ..., p_L) \neq 0$ and a number r such that $p \cdot z \geq r$ for every $z \in V$ and $p \cdot z \leq r$ for every $z \in Y + \{\bar{\omega}\}$.

Step 2: The sets V and $Y + \{\bar{\omega}\}$ are convex.

Step 3: $V \cap (Y + \{\bar{\omega}\}) = \emptyset$.

This follows from Pareto optimality. Suppose there was $z \in V \cap (Y + \{\bar{\omega}\})$. This means that there exists $x_i \in V_i$ and $y_j \in Y_j$ such that

$$z = \sum_i x_i = \sum_j y_j + \omega_j,$$

i.e., (x, y) is a Pareto improvement over (x^*, y^*), a contradiction.

Step 4: There is $p = (p_1, ..., p_L) \neq 0$ and a number r such that $p \cdot z \geq r$ for every $z \in V$ and $p \cdot z \leq r$ for every $z \in Y + \{\bar{\omega}\}$.

Given steps 2 and 3, this follows directly from the separating hyperplane theorem.

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By step 5, we have $p \cdot (\sum_i x_i^*) \geq r$. On the other hand, $\sum_i x_i^* = \sum_i y_j^* + \bar{\omega} \in Y + \{\bar{\omega}\}$, and therefore $p \cdot (\sum_i x_i^*) \leq r$. Thus, $p \cdot (\sum_i x_i^*) = r$. Since $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$, we also have $p \cdot (\bar{\omega} + \sum_j y_j^*) = r$.

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For any firm j and $y_j \in Y_j$, we have $y_j + \sum_{k \neq j} y_k^* \in Y$. Therefore, $p \cdot (\bar{\omega} + y_j + \sum_{h \neq j} y_h^*) \leq r = p \cdot (\bar{\omega} + y_j^* + \sum_{h \neq j} y_h^*)$. Hence, $p \cdot y_j \leq p \cdot y_j^*$.

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Step 8: For every i, if $x_i \succ_i x_i^*$, then $p \cdot x_i \geq p \cdot x_i^*$.

Consider any $x_i \succ_i x_i^*$. Because of steps 5 and 6, we have $p \cdot (x_i + \sum_{k \neq i} x_k^*) \geq r = p \cdot (x_i^* + \sum_{k \neq i} x_k^*)$. Hence, $p \cdot x_i \geq p \cdot x_i^*$.

Thus, given P.O. (x^*, y^*), we have found $p \neq 0$ such that

(i) For every j, $\quad p \cdot y_j \leq p \cdot y_j^* \quad$ for all $y_j \in Y_j$ (Step 7)

(ii) For every i, $\quad$ if $x_i \succ_i x_i^* \quad$ then $\quad p \cdot x_i \geq p \cdot x_i^*$ (Step 8)

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These are just the first two conditions of the definition of a price quasi equilibrium with transfers (if we let $w_i = p \cdot x_i^*$ for all i). Recall the definition: $\exists (w_i)$, with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$, s.t.

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Condition (iii) follows by hypothesis - (x^*, y^*) is a P.O. allocation.