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For each $(p, w) \gg 0$, the utility value of the UMP is denoted $v(p, w) \in \Re$. It is equal to $u(x^*)$ for any $x^* \in x(p, w)$. The function $v(p, w)$ is called the **indirect utility function**.

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Given $p \gg 0$ and $u > u(0)$, the value of the EMP is denoted $e(p, u)$. The function $e(p, u)$ is the **expenditure function.**

Proposition 3.E.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$ and that the price vector is $p \gg 0$. We have

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Proof: (i) (Uses LNS). Suppose that x^* is not optimal in the EMP with required utility level $u(x^*)$. Then there exists an x' such that $u(x') \geq u(x^*)$ and $p \cdot x' < p \cdot x^* \leq w$.

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By local nonsatiation, we can find an x'' very close to x' such that $u(x'') > u(x')$ and $p \cdot x'' < w$. But this implies that $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$, contradicting the optimality of x^* in the UMP.

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Thus, x^* must be optimal in the EMP when the required utility level is $u(x^*)$, and the minimized expenditure level is therefore $p \cdot x^*$.

Finally, since x^* solves the UMP when wealth is w, by Walras' law we have $p \cdot x^* = w$.

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Then there exists an x' such that $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Consider a bundle $x'' = \alpha x'$ where $\alpha \in (0, 1)$ (x'' is a "scaled-down" version of x').

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By continuity of $u(\cdot)$, if α is close enough to 1, then we will have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$.

But this contradicts the optimality of x^* in the EMP. Thus, x^* must be optimal in the UMP when wealth is $p \cdot x^*$, and the maximized utility level is therefore $u(x^*)$. In Proposition 3.E.3(ii), we will show that if x^* solves the EMP when the required utility level is u, then $u(x^*) = u$.

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Proof:

(i): homogeneity of degree 1. The constraint set of the EMP is unchanged when prices change. Thus, for any scalar $\alpha > 0$, minimizing $(\alpha p) \cdot x$ on this set leads to the same optimal consumption bundles as minimizing $p \cdot x$. Letting x^* be optimal in both circumstances, we have $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p, u)$.

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(ii): increasing in u non-decreasing in p_l. Suppose that $e(p, u)$ were not strictly increasing in u, and let x' and x'' denote optimal consumption bundles for required utility levels u' and u'', respectively, where $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$.

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Consider a bundle $\tilde{x} = \alpha x''$, where $\alpha \in (0, 1)$. By continuity of $u(\cdot)$, there exists an α close enough to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' > p \cdot \tilde{x}$. But this contradicts that x' being optimal in the EMP with required utility level u'.

To show that $e(p,u)$ is nondecreasing in p_l, suppose that price vectors p'' and p' have $p''_l \geq p'_l$ and $p''_k = p'_k$ for all $k \neq l$. Let x'' be an optimizing vector in the EMP for prices p''. Then $e(p'',u) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p',u)$, where the latter inequality follows from the definition of $e(p',u)$.

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where the last inequality follows because $u(x'') \geq \bar{u}$ and the definition of the expenditure function imply that $p \cdot x'' \geq e(p,\bar{u})$ and $p' \cdot x'' \geq e(p',\bar{u})$

Proposition 3.E.3: Suppose that $u(\cdot)$ is a continuous function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

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Proof: straightforward.

Proposition 3.G.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. For all p and u, the Hicksian demand $h(p,u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p,u) = \nabla_p e(p,u).$$

That is, $h_l(p,u) = \partial e(p,u)/\partial p_l$ for all $l = 1, ..., L$.

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Thus, given the expenditure function, we can calculate the consumer's Hicksian demand function simply by differentiating.

Proof 1: (Duality Theorem Argument). Since the expenditure function is precisely the support function for the set $K = \{x \in \Re^L_+ : u(x) \geq u\}$, and since the optimizing vector associated with this support function is $h(p,u)$, Proposition 3.F.1 implies that $h(p,u) = \nabla_p e(p,u)$.

Just as the derivatives of the utility function $u(\cdot)$ with respect to quantities have a price interpretation (we have seen in section 3.D that at an optimum they are equal to prices multiplied by a constant factor of proportionality), (3.G.1) tells us that the derivatives of the expenditure function $e(\cdot, u)$ with respect to prices have a quantity interpretation (they are equal to the Hicksian demands).

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Substituting from the first-order conditions for an interior solution to the EMP, $p = \lambda \nabla u(h(p,u))$, yields $\nabla_p e(p,u) = h(p,u) + \lambda[\nabla u(h(p,u)) \cdot D_p h(p,u)]^T$.

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Using the chain rule, the change in expenditure can be written as

$$\nabla_p e(p,u) = \nabla_p[p \cdot h(p,u)] = h(p,u) + [p \cdot D_p h(p,u)]^T.$$

Substituting from the first-order conditions for an interior solution to the EMP, $p = \lambda \nabla u(h(p,u))$, yields $\nabla_p e(p,u) = h(p,u) + \lambda[\nabla u(h(p,u)) \cdot D_p h(p,u)]^T$.

But since the constraint $u(h(p,u)) = u$ holds for all p in the EMP, we know that $\nabla u(h(p,u)) \cdot D_p h(p,u) = 0$, and so we have the result.

Proof 3: (Envelope Theorem Argument). Under the same simplifying assumptions used in Proof 2, we can directly appeal to the envelope theorem. Consider the value function $\phi(\alpha)$ of the constrained minimization problem:

$$\min_x f(x, \alpha) \qquad s.t. \quad g(x, \alpha) = 0.$$

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If $x^*(\alpha)$ is the (differentiable) solution to this problem as a function of the parameters $\alpha = (\alpha_1, ...\alpha_M)$, then the envelope theorem tells us that at any $\bar{\alpha} = (\bar{\alpha}_1, ...\bar{\alpha}_M)$ we have $\frac{\partial \phi(\bar{\alpha})}{\partial \alpha_m} = \frac{\partial f(x^*(\bar{\alpha}),\bar{\alpha})}{\partial \alpha_m} - \lambda \frac{\partial g(x^*(\bar{\alpha}),\bar{\alpha})}{\partial \alpha_m}$ for $m = 1, ..., M$, or in matrix notation, $\nabla_\alpha \phi(\bar{\alpha}) = \nabla_\alpha f(x^*(\bar{\alpha}), \bar{\alpha}) - \lambda \nabla_\alpha g(x^*(\bar{\alpha}), \bar{\alpha})$.

(See Section M.L of the Mathematical Appendix.) this result.

Because prices are parameters in the EMP that enter only the objective function $p \cdot x$, the change in the value function of the EMP with respect to a price change at $\bar{p}$, $\nabla_p e(\bar{p}, u)$, is just the vector of partial derivatives with respect to p of the objective function evaluated at the optimizing vector, $h(\bar{p}, u)$. Hence $\nabla_p e(p, u) = h(p, u)$.

Proposition 3.G.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

Proposition 3.G.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix by $D_p h(p, u)$. Then

(i) $D_p h(p, u) = D_p^2 e(p, u)$.

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(i) $D_p h(p, u) = D^2_p e(p, u)$.

(ii) $D_p h(p, u)$ is a negative semidefinite matrix.

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(i) $D_p h(p, u) = D_p^2 e(p, u)$.

(ii) $D_p h(p, u)$ is a negative semidefinite matrix.

(iii) $D_p h(p, u)$ is a symmetric matrix.

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Proof: Property (i) follows immediately from Proposition 3.G.1 by differentiation.

Properties (ii) and (iii) follow from property (i) and the fact that since $e(p, u)$ is a twice continuously differentiable concave function, it has a symmetric and negative semidefinite Hessian (i.e. second derivative) matrix (see Section M.C of the Mathematical Appendix).

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For property (iv), note that because $h(p,u)$ is homogeneous of degree zero in p, $h(\alpha p,u)-h(p,u) = 0$ for all α; differentiating this expression with respect to α yields $D_p h(p,u)p = 0$. [Note that because $h(p,u)$ is homogeneous of degree zero, $D_p h(p,u)p = 0$ also follows directly from Euler's formula; see Section M.B of the Mathematical Appendix.]

Proposition 3.G.3: (The Slutsky equation) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Then for all (p, w), and $u = v(p, w)$ we have

Proposition 3.G.3: (The Slutsky equation) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Then for all (p, w), and $u = v(p, w)$ we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \text{ for all } l, k \qquad (3.G.3)$$

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or equivalently, in matrix notation

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w)\ x(p,w)^T.$$

Proof: Consider a consumer facing the price-wealth pair $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$. Note that her wealth level $\bar{w}$ must satisfy $\bar{w} = e(\bar{p}, \bar{u})$. From condition (3.E.4), we know that for all (p, u), $h_l(p, u) = x_l(p, e(p, u))$. Differentiating this expression with respect to p_k and evaluating it at $(\bar{p}, \bar{u})$, we get

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}$$

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Finally, since $\bar{w} = e(\bar{p}, \bar{u})$ and $h_k(\bar{p}, \bar{u}) = x_k(\bar{p}, e(\bar{p}, \bar{u})) = x_k(\bar{p}, \bar{w})$, we have

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Propositon 3.G.4: (Roy's Identity). Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \Re^L_+$. Suppose also that the indirect utility function is differentialble at $(\bar{p}, \bar{w}) >> 0$. Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}).$$

That is, for every $l = 1, ..., L$: $x_l(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p}, \bar{w})/\partial p_l}{\partial v(\bar{p}, \bar{w})/\partial w}$.

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Proof 1: Let $u = v(\bar{p}, \bar{w}))$. Because $v(p, e(p, \bar{u})) = \bar{u}$ for all p, differentiating with respect to p and evaluating at $p = \bar{p}$:

$$\nabla_p v(\bar{p}, e(\bar{p}, \bar{u})) + \frac{\partial v(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \nabla_p e(\bar{p}, \bar{u}) = 0.$$

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But $\nabla_p e(\bar{p}, \bar{u}) = h(\bar{p}, \bar{u})$, and so:

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Proof 2: (First Order condition Argument). Assume that $x(p, w)$ is differentiable and $x(\bar{p}, \bar{w}) >> 0$. By the chain rule,

$$\frac{\partial v(\bar{p}, \bar{w})}{\partial p_l} = \sum_{k=1}^{L} \frac{\partial u(x(\bar{p}, \bar{w}))}{\partial x_k} \frac{\partial x_k(\bar{p}, \bar{w})}{\partial p_l}.$$

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$$\frac{\partial v(\bar{p}, \bar{w})}{\partial p_l} = \sum_{k=1}^{L} \lambda p_k \frac{\partial x_k(\bar{p}, \bar{w})}{\partial p_l} = -\lambda x_l(\bar{p}, \bar{w}),$$

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Finally, recall that $\lambda = \partial v(\bar{p}, \bar{w})/\partial w$.

Proof 3: (Envelope Theorem Argument). Applied to the UMP, the envelope theorem tells us directly that the utility effect of a marginal change in p_l is equal to its effect on the conusmer's budget constraint weighted by the Lagrange multiplier λ of the consumer's wealth constraint. That is,

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Similarly, the utility effect of a differential change in wealth $\partial v(p, w)/\partial w$ is just λ.