

## Existence of a Walrasian Equilibrium

As soon as an equilibrium state is defined for a model of an economy, the fundamental question of its existence is raised. [Gerard Debreu](#)

**Theorem 1** (*Brouwer's Fixed Point Theorem*) Suppose  $f : S \mapsto S$  is a continuous function, and  $S$  is a non-empty, compact, convex subset of  $R^l$ . The  $f$  has a fixed point, i.e., there exists  $\bar{x} \in S$  such that  $f(\bar{x}) = \bar{x}$ .

Consider an exchange economy in which each consumer has a well-defined demand function  $x_i(\cdot)$ .

A free disposal Walrasian equilibrium consist of  $(x_i(p), p)$  such that

$$\sum_i x_i(p) \leq \bar{\omega}.$$

Let the unit simplex in  $\mathfrak{R}^L$  be denoted  $\Delta$ :

$$\Delta = \{z \in \mathfrak{R}_+^L \mid \sum_i z_i = 1\}.$$

We will take  $\Delta$  to be the space of prices.

**Theorem 2** *Suppose  $x_i(\cdot)$  is a well defined, continuous function for all  $i$ . Then there exists a free disposal Walrasian equilibrium.*

**Proof:** Let  $x : \Delta \mapsto \sum_i X_i$  be defined as  $x(p) = \sum_i x_i(p)$ . As the sum of well defined, continuous functions,  $x$  is a well defined, continuous function.

Since  $x_i(\cdot)$  is a demand function for each  $i$ , it must satisfy the budget constraint. The aggregate demand function  $x$  must, therefore, satisfy the Walras's Law:

$$\text{for all } p \in \Delta, \quad p \cdot x(p) \leq p \cdot \bar{\omega}. \quad (1)$$

To show that a free disposal Walrasian equilibrium exists, what we have to find is some  $p \in \Delta$  for which  $(x(p) - \bar{\omega}) \leq 0$ .

Define a function  $f : \Delta \mapsto \mathfrak{R}^L$ ,

$$f(p) = p + (x(p) - \bar{\omega}).$$

If this function has a fixed point, say  $\bar{p}$ , then

$$\bar{p} = f(\bar{p}) = \bar{p} + (x(\bar{p}) - \bar{\omega}).$$

Of course, this means that  $(x(\bar{p}), \bar{p})$  is an equilibrium.

But the **only** problem is that although  $f$  is continuous we can't be sure that it has a fixed point. Brouwer's fixed point theorem can't be applied - because the range of  $f$  is not in  $\Delta$ .

Define the function  $g : \Delta \mapsto \Delta$  as

$$g(p)_h = \frac{p_h + \max(x(p)_h - \bar{\omega}_h, 0)}{1 + \sum_{j=1}^l \max(x(p)_j - \bar{\omega}_j, 0)} \quad h = 1, \dots, l. \quad (2)$$

Notice that for all  $p \in \Delta$ ,  $g(p)_h \geq 0$  for all  $h$  and  $\sum_h g(p)_h = 1$ . So  $g$  does map into  $\Delta$ .

Clearly,  $g$  is continuous in  $p$  and  $x$ . Since  $x$  is, by assumption, a continuous function of  $p$ , we can assert that  $g$  is a continuous function of  $p$ . Suppose we could find a  $p \in \Delta$  for which  $g(p) = p$ . Then it will turn out that at such a  $p$ ,  $\max(x(p)_h - \bar{\omega}_h, 0) = 0$  for all  $h$ . That clearly will be enough.

Since  $g$  is a continuous function from a non-empty, compact, convex set to itself, by Brouwer's fixed point Theorem, it has a fixed point, i.e., there exists  $\bar{p} \in \Delta$  such that  $g(\bar{p}) = \bar{p}$ . Now we can rewrite (2) at  $\bar{p}$  as

$$\bar{p}_h \sum_{j=1}^l \max(x(\bar{p})_j - \bar{\omega}_j, 0) = \max(x(\bar{p})_h - \bar{\omega}_h, 0) \quad \text{for all } h = 1, \dots, l. \quad (3)$$

We now claim that

$$x(\bar{p})_h \leq \bar{\omega}_h \quad \text{for all } h = 1, \dots, l. \quad (4)$$

Suppose (4) does not hold, i.e., there exists some commodity  $h$  for which  $x(\bar{p})_h > \bar{\omega}_h$ . Then it follows that  $\sum_{j=1}^l \max(x(\bar{p})_j - \bar{\omega}_j, 0) > 0$ . Now (3) implies that for every commodity  $h$  for which  $\bar{p}_h > 0$ ,  $x(\bar{p})_h > \bar{\omega}_h$ . But this means that  $\bar{p} \cdot x(\bar{p}) > \bar{p} \cdot \bar{\omega}$ , which contradicts (1). This establishes (4) and completes the proof. ■

There are two problems with this existence result.

1. As a general principle, we should make assumptions on the primitive data of the model, i.e., on the economy  $\mathcal{E} = \{(X_i, \succeq_i, \omega_i)\}$ . This is not the case with Theorem 2, which simply assumes that the demand functions are continuous.

What are sufficient conditions for  $x_i(\cdot)$  to be a well defined continuous function?

2. Demands are generally not single-valued - need to consider demand correspondences. This is especially important with production - CRS production sets have supply correspondences, not functions.

Suppose preferences are represented by a continuous utility function. Then we know that for each  $i$  and  $p \in \Delta$ ,  $x_i(p)$  is non-empty if  $X_i$  is compact. [We don't want to assume  $X_i$  to be compact, in fact,  $\mathbb{R}_+^L$  is not compact, so this will have to be dropped - eventually.] Further, we also know that  $x_i$  is a function if preferences are **strictly** convex. Under these assumptions,  $x_i$  is a continuous function of  $p$  if wealth is positive and fixed (in fact, it is enough to assume that  $\omega_i \gg 0$ ).

To consider the case of correspondences, we need a more general notion of continuity and a more general fixed point theorem.

We restrict attention to exchange economies.

A correspondence  $\phi : S \mapsto T$  is **upper hemicontinuous** at  $x \in \Delta$  if for every sequence  $x^q \rightarrow x$ ,  $x^q \in S$  for all  $q$  and every sequence  $y^q \rightarrow y$ , where  $y^q \in \phi(x^q)$  for all  $q$

$$y \in \phi(x).$$

**Lemma 1** *Suppose  $X_i$  is non-empty, convex and compact,  $\omega_i \gg 0$  and preferences are continuous and convex. Then  $x_i$  is a non-empty, convex-valued and upper hemicontinuous correspondence from  $\Delta$  to  $X_i$ .*

This allows us to prove existence by using Kakutani's fixed point theorem.

**Theorem 3** (*Kakutani's Fixed Point Theorem*) Suppose  $\phi : S \mapsto S$  is a non-empty, convex-valued and upper hemicontinuous correspondence, and  $S$  is a non-empty, compact, convex subset of  $\mathbb{R}^L$ . Then  $\phi$  has a fixed point, i.e., there exists  $\bar{x} \in S$  such that  $\bar{x} \in \phi(\bar{x})$ .

**Theorem 4** Under the assumptions of Lemma 1, there exists a free disposal Walrasian equilibrium.

**Proof:** By Lemma 1,  $x_i(\cdot)$  is non-empty, convex-valued and upper hemicontinuous for all  $i$ . Let  $X = \sum_i X_i$ . Since the sum of such mappings retains these properties, the aggregate demand correspondence,  $x = \sum_i x_i : \Delta \mapsto X$ , is non-empty, convex-valued and upper hemicontinuous. Define  $\gamma : X \mapsto \Delta$  as follows:

$$\gamma(x) = \{p \in \Delta \mid p \cdot (x - \bar{\omega}) \geq p' \cdot (x - \bar{\omega}) \text{ for all } p' \in \Delta\}.$$

[Homework: Prove that  $\gamma$  is non-empty, convex-valued and uhc.]

Now define the mapping  $\phi : \Delta \times X \mapsto \Delta \times X$ , where

$$\phi(p, x) = \gamma(x) \times x(p).$$

This is clearly, non-empty, convex-valued and upper hemicontinuous (since it is a product of such mappings).

By Kakutani's fixed point theorem,  $\phi$  has a fixed point,  $(\bar{p}, \bar{x})$ .

Note that  $(\bar{p}, \bar{x}) \in \phi(\bar{p}, \bar{x})$  means that

$$\bar{p} \in \gamma(\bar{x}), \quad \bar{x} \in x(\bar{p}).$$

The latter condition means that there exist  $\bar{x}_i \in x_i(\bar{p})$  such that  $\sum_i \bar{x}_i = \bar{x}$ . We claim that  $(\bar{p}, (\bar{x}_i))$  is a free disposal Walrasian equilibrium.

We have to show that  $\bar{x} \leq \bar{\omega}$ .

By Walras' Law,  $\bar{p} \cdot (\bar{x} - \bar{\omega}) \leq 0$ .

From the construction of  $\gamma$ , and the fact that  $\bar{p} \in \gamma(\bar{x})$ , it follows that

$$p \cdot (\bar{x} - \bar{\omega}) \leq 0 \text{ for all } p \in \Delta.$$

In particular,  $p^j \cdot (\bar{x} - \bar{\omega}) \leq 0$ , where  $p^j \in \Delta$  is defined such that all coordinates other than  $j$  are 0 and  $p_j^j = 1$ . But then,  $p^j \cdot (\bar{x} - \bar{\omega}) = \bar{x}_j - \bar{\omega}_j \leq 0$ .

The same argument can be repeated for other coordinates, to claim that  $\bar{x} - \bar{\omega} \leq 0$ . This completes the proof. ■