Existence of a Walrasian Equilibrium

Existence of a Walrasian Equilibrium

As soon as an equilibrium state is defined for a model of an economy, the fundamental question of its existence is raised. Gerard Debreu

Existence of a Walrasian Equilibrium

As soon as an equilibrium state is defined for a model of an economy, the fundamental question of its existence is raised. Gerard Debreu

Theorem 1 *(Brouwer's Fixed Point Theorem) Suppose* $f : S \mapsto S$ *is a continuous function, and* S *is a non-empty, compact, convex subset of* R^l*. The* f *has a fixed point, i.e., there exists* $\bar{x} \in S$ *such that* $f(\bar{x}) = \bar{x}$*.*

Consider an exchange economy in which each consumer has a well-defined demand function $x_i(.)$.

A free disposal Walrasian equilibrium consist of $(x_i(p,),p)$ such that

$$\sum_i x_i(p) \leq \bar{\omega}.$$

Let the unit simplex in $\Re^L$ be denoted $\triangle$:

$$\triangle = \{z \in \Re^L_+ | \sum_i z_i = 1\}.$$

We will take $\triangle$ to be the space of prices.

Theorem 2 *Suppose* $x_i(.)$ *is a well defined, continuous function for all* i*. Then there exists a free disposal Walrasian equilibrium.*

Theorem 2 *Suppose $x_i(.)$ is a well defined, continuous function for all i. Then there exists a free disposal Walrasian equilibrium.*

Proof: Let $x : \triangle \mapsto \sum_i X_i$ be defined as $x(p) = \sum_i x_i(p)$. As the sum of well defined, continuous functions, x is a well defined, continuous function.

Theorem 2 *Suppose* $x_i(.)$ *is a well defined, continuous function for all* i*. Then there exists a free disposal Walrasian equilibrium.*

Proof: Let $x : \triangle \mapsto \sum_i X_i$ be defined as $x(p) = \sum_i x_i(p)$. As the sum of well defined, continuous functions, x is a well defined, continuous function.

Since $x_i(.)$ is a demand function for each i, it must satisfy the budget constraint. The aggregate demand function x must, therefore, satisfy the Walras's Law:

$$\text{for all } p \in \triangle, \quad p \cdot x(p) \leq p \cdot \bar{\omega}. \tag{1}$$

To show that a free disposal Walrasian equilibrium exists, what we have to find is some $p \in \triangle$ for which $(x(p) - \bar{\omega}) \leq 0$.

To show that a free disposal Walrasian equilibrium exists, what we have to find is some $p \in \triangle$ for which $(x(p) - \bar{\omega}) \leq 0$.

Define a function $f : \triangle \mapsto \Re^L$,

$$f(p) = p + (x(p) - \bar{\omega}).$$

To show that a free disposal Walrasian equilibrium exists, what we have to find is some $p \in \triangle$ for which $(x(p) - \bar{\omega}) \leq 0$.

Define a function $f : \triangle \mapsto \Re^L$,

$$f(p) = p + (x(p) - \bar{\omega}).$$

If this function has a fixed point, say $\bar{p}$, then

$$\bar{p} = f(\bar{p}) = \bar{p} + (x(\bar{p}) - \bar{\omega}).$$

To show that a free disposal Walrasian equilibrium exists, what we have to find is some $p \in \triangle$ for which $(x(p) - \bar{\omega}) \leq 0$.

Define a function $f : \triangle \mapsto \Re^L$,

$$f(p) = p + (x(p) - \bar{\omega}).$$

If this function has a fixed point, say $\bar{p}$, then

$$\bar{p} = f(\bar{p}) = \bar{p} + (x(\bar{p}) - \bar{\omega}).$$

Of course, this means that $(x(\bar{p}), \bar{p})$ is an equilibrium.

To show that a free disposal Walrasian equilibrium exists, what we have to find is some $p \in \triangle$ for which $(x(p) - \bar{\omega}) \leq 0$.

Define a function $f : \triangle \mapsto \Re^L$,

$$f(p) = p + (x(p) - \bar{\omega}).$$

If this function has a fixed point, say $\bar{p}$, then

$$\bar{p} = f(\bar{p}) = \bar{p} + (x(\bar{p}) - \bar{\omega}).$$

Of course, this means that $(x(\bar{p}), \bar{p})$ is an equilibrium.

But the only problem is that although f is continuous we can't be sure that it has a fixed point. Brouwer's fixed point theorem can't be applied - because the range of f is not in $\triangle$.

Define the function $g : \triangle \mapsto \triangle$ as

$$g(p)_h = \frac{p_h + \max(x(p)_h - \bar{\omega}_h, 0)}{1 + \sum_{j=1}^{l} \max(x(p)_j - \bar{\omega}_j, 0)} \qquad h = 1, \ldots, l. \qquad (2)$$

Define the function $g : \triangle \mapsto \triangle$ as

$$g(p)_h = \frac{p_h + \max(x(p)_h - \bar{\omega}_h, 0)}{1 + \sum_{j=1}^{l} \max(x(p)_j - \bar{\omega}_j, 0)} \qquad h = 1, \ldots, l. \qquad (2)$$

Notice that for all $p \in \triangle$, $g(p)_h \geq 0$ for all h and $\sum_h g(p)_h = 1$. So g does map into $\triangle$.

Define the function $g : \triangle \mapsto \triangle$ as

$$g(p)_h = \frac{p_h + \max(x(p)_h - \bar{\omega}_h, 0)}{1 + \sum_{j=1}^{l} \max(x(p)_j - \bar{\omega}_j, 0)} \qquad h = 1, \ldots, l. \qquad (2)$$

Notice that for all $p \in \triangle$, $g(p)_h \geq 0$ for all h and $\sum_h g(p)_h = 1$. So g does map into $\triangle$.

Clearly, g is continuous in p and x. Since x is, by assumption, a continuous function of p, we can assert that g is a continuous function of p. Suppose we could find a $p \in \triangle$ for which $g(p) = p$. Then it will turn out that at such a p, $\max(x(p)_h - \bar{\omega}_h, 0) = 0$ for all h. That clearly will be enough.

Since g is a continuous function from a non-empty, compact, convex set to itself, by Brouwer's fixed point Theorem, it has a fixed point, i.e., there exists $\bar{p} \in \triangle$ such that $g(\bar{p}) = \bar{p}$. Now we can rewrite (2) at $\bar{p}$ as

$$\bar{p}_h \sum_{j=1}^{l} \max(x(\bar{p})_j - \bar{\omega}_j, 0) = \max(x(\bar{p})_h - \bar{\omega}_h, 0) \quad \text{for all } h = 1, \ldots, l. \tag{3}$$

Since g is a continuous function from a non-empty, compact, convex set to itself, by Brouwer's fixed point Theorem, it has a fixed point, i.e., there exists $\bar{p} \in \triangle$ such that $g(\bar{p}) = \bar{p}$. Now we can rewrite (2) at $\bar{p}$ as

$$\bar{p}_h \sum_{j=1}^{l} \max(x(\bar{p})_j - \bar{\omega}_j, 0) = \max(x(\bar{p})_h - \bar{\omega}_h, 0) \quad \text{for all } h = 1, \ldots, l. \tag{3}$$

We now claim that

$$x(\bar{p})_h \leq \bar{\omega}_h \text{ for all } h = 1, \ldots, l. \tag{4}$$

Since g is a continuous function from a non-empty, compact, convex set to itself, by Brouwer's fixed point Theorem, it has a fixed point, i.e., there exists $\bar{p} \in \triangle$ such that $g(\bar{p}) = \bar{p}$. Now we can rewrite (2) at $\bar{p}$ as

$$\bar{p}_h \sum_{j=1}^{l} \max(x(\bar{p})_j - \bar{\omega}_j, 0) = \max(x(\bar{p})_h - \bar{\omega}_h, 0) \quad \text{for all } h = 1, \ldots, l. \tag{3}$$

We now claim that

$$x(\bar{p})_h \leq \bar{\omega}_h \text{ for all } h = 1, \ldots, l. \tag{4}$$

Suppose (4) does not hold, i.e., there exists some commodity h for which $x(\bar{p})_h > \bar{\omega}_h$. Then it follows that $\sum_{j=1}^{l} \max(x(\bar{p})_j - \bar{\omega}_j, 0) > 0$. Now (3) implies that for every commodity h for which $\bar{p}_h > 0$, $x(\bar{p})_h > \bar{\omega}_h$. But this means that $\bar{p} \cdot x(\bar{p}) > \bar{p} \cdot \bar{\omega}$, which contradicts (1). This establishes (4) and completes the proof. ■

There are two problems with this existence result.

There are two problems with this existence result.

1. As a general principle, we should make assumptions on the primitive data of the model, i.e., on the economy $\mathcal{E} = \{(X_i, \succeq_i , \omega_i)\}$. This is not the case with Theorem 2, which simply assumes that the demand functions are continuous.

There are two problems with this existence result.

1. As a general principle, we should make assumptions on the primitive data of the model, i.e., on the economy $\mathcal{E} = \{(X_i, \succeq_i, \omega_i)\}$. This is not the case with Theorem 2, which simply assumes that the demand functions are continuous.

   What are sufficient conditions for $x_i(.)$ to be a well defined continuous function?

There are two problems with this existence result.

1. As a general principle, we should make assumptions on the primitive data of the model, i.e., on the economy $\mathcal{E} = \{(X_i, \succeq_i, \omega_i)\}$. This is not the case with Theorem 2, which simply assumes that the demand functions are continuous.

   What are sufficient conditions for $x_i(.)$ to be a well defined continuous function?

2. Demands are generally not single-valued - need to consider demand correspondences. This is especially important with production - CRS production sets have supply correspondences, not functions.

Suppose preferences are represented by a continuous utility function. Then we know that for each i and $p \in \triangle$, $x_i(p)$ is non-empty if X_i is compact. [We don't want to assume X_i to be compact, in fact, $\Re_+^L$ is not compact, so this will have to be dropped - eventually.] Further, we also know that x_i is a function if preferences are strictly convex. Under these assumptions, x_i is a continuous function of p if wealth is positive and fixed (in fact, it is enough to assume that $\omega_i \gg 0$).

Suppose preferences are represented by a continuous utility function. Then we know that for each i and $p \in \triangle$, $x_i(p)$ is non-empty if X_i is compact. [We don't want to assume X_i to be compact, in fact, $\Re^L_+$ is not compact, so this will have to be dropped - eventually.] Further, we also know that x_i is a function if preferences are strictly convex. Under these assumptions, x_i is a continuous function of p if wealth is positive and fixed (in fact, it is enough to assume that $\omega_i \gg 0$).

To consider the case of correspondences, we need a more general notion of continuity and a more general fixed point theorem.

Suppose preferences are represented by a continuous utility function. Then we know that for each i and $p \in \triangle$, $x_i(p)$ is non-empty if X_i is compact. [We don't want to assume X_i to be compact, in fact, $\Re^L_+$ is not compact, so this will have to be dropped - eventually.] Further, we also know that x_i is a function if preferences are strictly convex. Under these assumptions, x_i is a continuous function of p if wealth is positive and fixed (in fact, it is enough to assume that $\omega_i \gg 0$).

To consider the case of correspondences, we need a more general notion of continuity and a more general fixed point theorem.

We restrict attention to exchange economies.

A correspondence $\phi : S \mapsto T$ is upper hemicontinuous at $x \in \triangle$ if for every sequence $x^q \to x$, $x^q \in S$ for all q and every sequence $y^q \to y$, where $y^q \in \phi(x^q)$ for all q

$$y \in \phi(x).$$

A correspondence $\phi : S \mapsto T$ is upper hemicontinuous at $x \in \triangle$ if for every sequence $x^q \rightarrow x$, $x^q \in S$ for all q and every sequence $y^q \rightarrow y$, where $y^q \in \phi(x^q)$ for all q

$$y \in \phi(x).$$

Lemma 1 *Suppose X_i is non-empty, convex and compact, $\omega_i \gg 0$ and preferences are continuous and convex. Then x_i is a non-empty, convex-valued and upper hemicontinuous correspondence from $\triangle$ to X_i.*

A correspondence $\phi : S \mapsto T$ is upper hemicontinuous at $x \in \triangle$ if for every sequence $x^q \rightarrow x$, $x^q \in S$ for all q and every sequence $y^q \rightarrow y$, where $y^q \in \phi(x^q)$ for all q

$$y \in \phi(x).$$

Lemma 1 *Suppose X_i is non-empty, convex and compact, $\omega_i \gg 0$ and preferences are continuous and convex. Then x_i is a non-empty, convex-valued and upper hemicontinuous correspondence from $\triangle$ to X_i.*

This allows us to prove existence by using Kakutani's fixed point theorem.

Theorem 3 *(Kakutani's Fixed Point Theorem) Suppose* $\phi : S \mapsto S$ *is a non-empty, convex-valued and upper hemicontinuous correspondence, and* S *is a non-empty, compact, convex subset of* $\Re^L$. *Then* ϕ *has a fixed point, i.e., there exists* $\bar{x} \in S$ *such that* $\bar{x} \in \phi(\bar{x})$.

Theorem 3 *(Kakutani's Fixed Point Theorem) Suppose* $\phi : S \mapsto S$ *is a non-empty, convex-valued and upper hemicontinuous correspondence, and* S *is a non-empty, compact, convex subset of* $\Re^L$*. Then* ϕ *has a fixed point, i.e., there exists* $\bar{x} \in S$ *such that* $\bar{x} \in \phi(\bar{x})$*.*

Theorem 4 *Under the assumptions of Lemma 1, there exists a free disposal Walrasian equilibrium.*

Theorem 3 *(Kakutani's Fixed Point Theorem) Suppose* $\phi : S \mapsto S$ *is a non-empty, convex-valued and upper hemicontinuous correspondence, and* S *is a non-empty, compact, convex subset of* $\Re^L$*. Then* ϕ *has a fixed point, i.e., there exists* $\bar{x} \in S$ *such that* $\bar{x} \in \phi(\bar{x})$*.*

Theorem 4 *Under the assumptions of Lemma 1, there exists a free disposal Walrasian equilibrium.*

Proof: By Lemma 1, $x_i(.)$ is non-empty, convex-valued and upper hemicontinuous for all i. Let $X = \sum_i X_i$. Since the sum of such mappings retains these properties, the aggregate demand correspondence, $x = \sum_i x_i : \triangle \mapsto X$, is non-empty, convex-valued and upper hemicontinuous. Define $\gamma : X \mapsto \triangle$ as follows:

$$\gamma(x) = \{p \in \triangle \mid p \cdot (x - \bar{\omega}) \geq p' \cdot (x - \bar{\omega}) \text{ for all } p' \in \triangle\}.$$

Theorem 3 *(Kakutani's Fixed Point Theorem) Suppose* $\phi : S \mapsto S$ *is a non-empty, convex-valued and upper hemicontinuous correspondence, and* S *is a non-empty, compact, convex subset of* $\Re^L$*. Then* ϕ *has a fixed point, i.e., there exists* $\bar{x} \in S$ *such that* $\bar{x} \in \phi(\bar{x})$*.*

Theorem 4 *Under the assumptions of Lemma 1, there exists a free disposal Walrasian equilibrium.*

Proof: By Lemma 1, $x_i(.)$ is non-empty, convex-valued and upper hemicontinuous for all i. Let $X = \sum_i X_i$. Since the sum of such mappings retains these properties, the aggregate demand correspondence, $x = \sum_i x_i : \triangle \mapsto X$, is non-empty, convex-valued and upper hemicontinuous. Define $\gamma : X \mapsto \triangle$ as follows:

$$\gamma(x) = \{p \in \triangle \mid p \cdot (x - \bar{\omega}) \geq p' \cdot (x - \bar{\omega}) \text{ for all } p' \in \triangle\}.$$

[Homework: Prove that γ is non-empty, convex-valued and uhc.]

Now define the mapping $\phi : \triangle \times X \mapsto \triangle \times X$, where

$$\phi(p, x) = \gamma(x) \times x(p).$$

This is clearly, non-empty, convex-valued and upper hemicontinuous (since it is a product of such mappings).

Now define the mapping $\phi : \triangle \times X \mapsto \triangle \times X$, where

$$\phi(p, x) = \gamma(x) \times x(p).$$

This is clearly, non-empty, convex-valued and upper hemicontinuous (since it is a product of such mappings).

By Kakutani's fixed point theorem, ϕ has a fixed point, $(\bar{p}, \bar{x})$.

Now define the mapping $\phi : \triangle \times X \mapsto \triangle \times X$, where

$$\phi(p, x) = \gamma(x) \times x(p).$$

This is clearly, non-empty, convex-valued and upper hemicontinuous (since it is a product of such mappings).

By Kakutani's fixed point theorem, ϕ has a fixed point, $(\bar{p}, \bar{x})$.

Note that $(\bar{p}, \bar{x}) \in \phi(\bar{p}, \bar{x})$ means that

$$\bar{p} \in \gamma(\bar{x}), \qquad \bar{x} \in x(\bar{p}).$$

Now define the mapping $\phi : \triangle \times X \mapsto \triangle \times X$, where

$$\phi(p, x) = \gamma(x) \times x(p).$$

This is clearly, non-empty, convex-valued and upper hemicontinuous (since it is a product of such mappings).

By Kakutani's fixed point theorem, ϕ has a fixed point, $(\bar{p}, \bar{x})$.

Note that $(\bar{p}, \bar{x}) \in \phi(\bar{p}, \bar{x})$ means that

$$\bar{p} \in \gamma(\bar{x}), \qquad \bar{x} \in x(\bar{p}).$$

The latter condition means that there exist $\bar{x}_i \in x_i(\bar{p})$ such that $\sum_i \bar{x}_i = \bar{x}$. We claim that $(\bar{p}, (\bar{x}_i))$ is a free disposal Walrasian equilibrium.

We have to show that $\bar{x} \leq \bar{\omega}$.

We have to show that $\bar{x} \leq \bar{\omega}$.

By Walras' Law, $\bar{p} \cdot (\bar{x} - \bar{\omega}) \leq 0$.

We have to show that $\bar{x} \leq \bar{\omega}$.

By Walras' Law, $\bar{p} \cdot (\bar{x} - \bar{\omega}) \leq 0$.

From the construction of γ, and the fact that $\bar{p} \in \gamma(\bar{x})$, it follows that

$$p \cdot (\bar{x} - \bar{\omega}) \leq 0 \text{ for all } p \in \triangle.$$

We have to show that $\bar{x} \leq \bar{\omega}$.

By Walras' Law, $\bar{p} \cdot (\bar{x} - \bar{\omega}) \leq 0$.

From the construction of γ, and the fact that $\bar{p} \in \gamma(\bar{x})$, it follows that

$$p \cdot (\bar{x} - \bar{\omega}) \leq 0 \text{ for all } p \in \triangle.$$

In particular, $p^j \cdot (\bar{x} - \bar{\omega}) \leq 0$, where $p^j \in \triangle$ is defined such that all coordinates other than j are 0 and $p^j_j = 1$. But then, $p^j \cdot (\bar{x} - \bar{\omega}) = \bar{x}_j - \bar{\omega}_j \leq 0$.

We have to show that $\bar{x} \leq \bar{\omega}$.

By Walras' Law, $\bar{p} \cdot (\bar{x} - \bar{\omega}) \leq 0$.

From the construction of γ, and the fact that $\bar{p} \in \gamma(\bar{x})$, it follows that

$$p \cdot (\bar{x} - \bar{\omega}) \leq 0 \text{ for all } p \in \triangle.$$

In particular, $p^j \cdot (\bar{x} - \bar{\omega}) \leq 0$, where $p^j \in \triangle$ is defined such that all coordinates other than j are 0 and $p^j_j = 1$. But then, $p^j \cdot (\bar{x} - \bar{\omega}) = \bar{x}_j - \bar{\omega}_j \leq 0$.

The same argument can be repeated for other coordinates, to claim that $\bar{x} - \bar{\omega} \leq 0$. This completes the proof. ■