EC 206, Spring 2003
Outline of Answers to Mid-Term Exam

2. Preferences are strictly convex, and it therefore suffices to find some commodity bundle satisfying the Kuhn-Tucker first order conditions. When does there exist an interior solution? If there is a tangency on the budget line, i.e., if there exists a point on the budget line such that $\sqrt{x_2}/2 = p_1/p_2$, or $x_2 = 4p_1^2/p_2^2$. For this to be on the budget line, we need $4p_1^2/p_2 \leq w$. And in this case the demand is $([w - 4p_1^2/p_2]/p_1, 4p_1^2/p_2^2)$. If $w < 4p_1^2/p_2$, the maximum amount of x_2 that is affordable will lead to the MRS being lower than p_1/p_2. This implies that the entire income will be spent on commodity 2. The demand, then is $(0, w/p_2)$. Thus

$$x((p_1,p_2),w) = \begin{cases} (0, w/p_2) & \text{if } w < 4p_1^2/p_2 \\ ([w - 4p_1^2/p_2]/p_1, 4p_1^2/p_2^2) & \text{otherwise} \end{cases}$$

At $p = (1,1)$ and $w = 1$, demand is given by the first case in the above expression. Moreover, for all prices converging to $(1,1)$ and wealth converging to 1, we remain in the case where the entire wealth is spent on commodity 2. Since w/p_2 is a continuous function, demand is continuous at these prices and wealth.

If commodity 2 can only be consumed in integer amounts, demand for it must be 0 whenever $w/p_2 < 1$. However, demand for commodity 2 is 1 at $((1,1),1)$. To show that demand is not continuous all we need to do is find prices and wealths converging to $((1,1),1)$ such that the point $(0,1)$ lies above the budget line. For example, define $w^k = w = 1$ for all k and $p^k = (1, 1 + 1/k)$. Clearly, $p^k \to (1,1)$, and

$$x(p^k, w^k) = (1,0) \text{ for all } k,$$

which means that $x(p^k, w^k) \to (1,0)$. However, $x(p,w) = (0,1)$.

3. Since the consumption set is closed, $x^* \in X$. $p^k \cdot x^k \leq w$ for all k, and $p.x$ is a continuous function of p and x, taking limits, we have $p \cdot x^* \leq w$. This shows that x^* is affordable at (p,w). Suppose there exists $x' \succ x^*$ such that $p \cdot x' \leq w$. There are now two possibilities:

(a) $x' > 0$. In this case, given that $p \gg 0$, $X = R^L_+$ and $w > 0$, we can find $\hat{x}$ arbitrarily close to x' such that $p \cdot \hat{x} < w$. For $\hat{x}$ close enough to x' we also know, by continuity of preferences, that $\hat{x} \succ x^*$. Thus we have found $\hat{x}$ such that:

$$\hat{x} \succ x^* \text{ and } p \cdot \hat{x} < w.$$

(b) $x' = 0$. In this case, because $w > 0$, the above is true for $\hat{x} = x'$.

Since $x^k \to x^*$, by continuity of preferences, $\hat{x} \succ x^k$ for k large enough. (This relies on worse-than sets to be open). Moreover, $p \cdot \hat{x} < w$ implies that for k large enough, we also have $p^k \cdot x^k < w$. Thus, there exists k large enough such that

$$\hat{x} \succ x^k \text{ and } p^k \cdot \hat{x} < w.$$

But this contradicts the hypothesis that x^k is the demand at (p^k, w).

4. Note that no assumptions have been made in the first part. And the result in the first part does not require *any* assumptions.

   At prices p and wealth w_i, x_i is consumer i's demand. This means that

$$\text{for all } i, \text{ if } u_i(x_i') > u_i(x_i), \text{ then } p \cdot x_i' > w_i. \tag{1}$$

   Of course, we also have, $p \cdot x_i \leq w_i$ for all i. To prove the result, argue by contradiction. Suppose there does exist (x_i') such that

$$u_i(x_i') > u_i(x_i) \text{ for all } i$$

   and

$$\sum_i x_i' = \bar{x} = \sum_i x_i. \tag{2}$$

   By (1), the first condition implies that $p \cdot x_i' > w_i$ for all i. Adding across all i, (and using the fact that $p \cdot x_i \leq w_i$ for all i) this yields,

$$p \cdot \sum_i x_i' > p \cdot \sum_i x_i. \tag{3}$$

   But this contradicts (2), and completes the proof.

Suppose we try to mimic this proof to show that there does not exist a Pareto improvement. Again, by contradiction, suppose there is (x_i') such that (2) holds,

$$u_i(x_i') \geq u_i(x_i) \text{ for all } i, \text{ and there exists } j \text{ such that } u_j(x_j') > u_j(x_j).$$

Of course, we can again claim that

$$p \cdot x_j' > p \cdot x_j.$$

Obviously, this strict inequality may not hold for other i. But if we could claim that

$$\text{for all } i, \text{ if } u_i(x_i') \geq u_i(x_i), \text{ then } p \cdot x_i' \geq w_i \tag{4}$$

then we would again obtain (3), and the proof would be complete. So what remains is to prove (4). Recall that we still have condition (1). When does (1) imply (4)? What we are asking for is the duality result that utility maximization implies 'expenditure minimization'. Local non-satiation will be sufficient. For if there exists x_i' such that $u_i(x_i') \geq u_i(x_i)$ and $p \cdot x_i' < w_i$, then we can find, in a neighborhood of x_i', some z_i such that $u_i(z_i) > u_i(x_i)$, and $p \cdot z_i < w_i$, contradicting (1). Monotonicity, which implies local non-satiation, will also suffice.

If local non-satiation does not hold, it is possible that x_i satisfies (1) but not (4). And then, it is possible that a Pareto improvement can be made (even though a weak Pareto improvement is impossible).