The Weak Axiom and Walrasian Demand

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The Walrasian demand function $x(p, w)$ satisfies the **Weak Axiom of Revealed Preference** (WARP) if for any two pairs (p, w) and (p', w') the following holds:

if $p \cdot x' \leq w$ and $x \neq x'$, then $p' \cdot x > w'$.

(p', w') is said to be a **compensated price change** from (p, w) if

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Proposition 2.F.1. Suppose $x(p, w)$ is homogeneous of degree 0 and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom iff the following property holds:

> For any compensated price change from (p, w) to (p', w'), we have
>
> $$(p' - p) \cdot (x' - x) \leq 0, \text{ with } < 0 \text{ if } x' \neq x. \qquad (2.F.1)$$

For a compensated price change:

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Proof:

(a) WARP implies (2.F.1).

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The result is obvious if $x = x'$, so suppose $x \neq x'$.

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The LHS of (2.F.1) can be rewritten as:

$$p' \cdot [x' - x] - p \cdot [x' - x].$$

To show: $p' \cdot [x' - x] - p \cdot [x' - x] < 0$.

Because the change to (p', w') is a compensated change, $p' \cdot x = w'$.

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By Walras' law, $p' \cdot x' = w'$, and so $p' \cdot [x' - x] = 0$. Thus the LHS of (2.F.1) is

$$-p \cdot [x' - x].$$

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Since x is affordable under (p', w'), by the weak axiom, it must be the case that x' is unaffordable under (p, x), i.e., $p \cdot x' > w$.

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By Walras' law, $p \cdot x = w$, and so we have

$$p \cdot [x' - x] > 0$$

which completes the proof that (2.F.1) holds.

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It suffices to show that if WARP fails, then there must be a compensated change for which it fails.

So let us suppose that there are two pairs (p', w') and (p'', w'') for which WARP fails. Let

$$x' \equiv x(p', w'), \qquad x'' \equiv x(p'', w'').$$

Since WARP fails for these two situations, we have:

$$x' \neq x", \quad p' \cdot x" \leq w', \quad p" \cdot x' \leq w".$$

If one of these inequalities holds as an equality, then we have a compensated price changes and the proof would be complete. So henceforth we can assume that

$$p' \cdot x" < w', \quad p" \cdot x' < w".$$

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$$p' \cdot x'' < w', \quad p'' \cdot x' < w'' .$$

Choose $\alpha \in (0, 1)$ such that

$$(\alpha p' + (1 - \alpha)p'') \cdot x' = (\alpha(p' + (1 - \alpha)p'') \cdot x'' ,$$

and denote $p = \alpha p' + (1 - \alpha)p''$ and $w = p \cdot x'$.

Now,

$$\alpha w' + (1-\alpha)w'' \quad > \alpha p' \cdot x' + (1-\alpha)p'' \cdot x'$$

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This implies that either $p' \cdot x(p,w) < w'$ or $p'' \cdot x(p,w) < w''$. Without loss of generality, assume the former.

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This implies that either $p' \cdot x(p,w) < w'$ or $p'' \cdot x(p,w) < w''$. Without loss of generality, assume the former.

But now we have a violation of WARP for the compensated price change from (p', w') to (p, w) because

$$x(p,w) \neq x', p \cdot x' = w \text{ and } p' \cdot x(p,w) < w'.$$

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More importantly, we have seen that it is equivalent to the weak axiom.

Now, suppose $x(p, w)$ is differentiable, and consider a differential change in prices, dp. A compensated change now means that $dw = x(p, w) \cdot dp$ and WARP says that

$$dp \cdot dx \leq 0 \qquad (2.F.5)$$

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$$dx = D_p x(p, w)dp + D_w x(p, w)[x(p, w) \cdot dp],$$

or

$$dx = [D_p x(p, w) + D_w x(p, w)x(p, w)^T]dp.$$

To this last expression, namely,

$$dx = [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp.$$

apply WARP, $dp \cdot dx \leq 0$, to get:

$$dp \cdot [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp \leq 0 \qquad (2.F.9)$$

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A typical element of $S(p,w)$ is:

$$s_{lk} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w).$$