Efficiency in a Public Goods Economy

Foley's Second Welfare Theorem and Lindahl Equilibrium

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m consumers and a single firm.

Each consumer has a consumption set $X_i = X_{\pi,i} \times X_g \subseteq R^{l+k}$ and a preference relation $\succeq_i$. The aggregate endowment is $\omega = (\omega_\pi, 0)$.

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The firm has a production set Y which is a closed convex cone with vertex 0 (constant returns to scale).

$((x_{\pi,i}), x_g, y)$ is an allocation if $(x_{\pi,i}, x_g) \in X_i$ for all i, $y \in Y$ and

$$\sum_i x_{\pi,i} = y_\pi + \omega_\pi,$$

$$x_g = y_g.$$

The Free Rider Problem in a Simple Model

One private good and one public good.

Public good produced under CRS at a marginal cost of c.

Let x_i denote $i's$ consumption of the private good and let y denote the common consumption of y. The feasibility condition is simply that

$$\sum_i x_i - c(y) = \omega.$$

If $((x_i), y)$ is Pareto optimal, then (modulo boundary conditions, and assuming differentiability)

$$\sum_i \frac{u_{i,y}(x_i, y)}{u_{ix_i}(x_i, y)} = c.$$

Now consider the case of quasi linear utility functions, $u_i(x_i, y) = x_i + v_i(y)$ and $c = 1$.

$$\frac{u_{i,y}(x_i, y)}{u_{ix_i}(x_i, y)} = v_{i,y}(y).$$

Thus, the optimality condition is:

$$\sum_i v_{i,y}(y) = 1.$$

Suppose

$$u_i(x_i, y) = x_i + \alpha_i \sqrt{y}.$$

In this case, Pareto optimality requires that the level of public good be such that

$$y = 0.25(\sum_i \alpha_i)^2.$$

Correspondingly, for each i,

$$v_{iy}(y) = \frac{\alpha_i}{2\sqrt{y}} = \frac{\alpha_i}{\sum_i \alpha_i}.$$

Volumtary Contributions

Suppose the government asks for voluntary contributions. Every dollar contributed is spent on producing the public good.

Consumer i will now take into account what is already contributed by the others and make a contribution to maximize his utility given the contribution of the others.

Suppose $m = 2$ and $\alpha_1 = \alpha_2 = 2$.

Suppose 1 contributes 0, what will 2 contribute?

Consumer 2 looks at an opportunity line that has slope -1 and starts from ω_2. He maximizes utility by choosing $y = 1$ and contributing one dollar.

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Consumer 1 can consume 1 unit of y simply because y is a public good and consumer 2 has paid for one unit of it. Thus consumer 1 can consume $(\omega_1, 1)$. To consume more of the public good, he must contribute an equivalent amount of the private good. Thus his opportunity line consists of the point $(\omega_1, 1)$ and has slope -1 for higher consumption levels of y. And it is optimal for 1 to contribute 0.

Thus the contributions 0 and 1 constitute an equilibrium.

There are other equilibria but they all yield an aggregate contribution of one unit.

Thus, with voluntary contributions the level of y is 1.

But the Pareto optimal level is 4.

With voluntary contributions too little of the public good is produced. This is known as the free rider problem.

The inefficiency of the voluntary contributions equilibrium can be explained as follows.

Any i who makes a contribution in equilibrium will ensure that his marginal rate of substitution equals 1. But this will mean that the sum of the marginal rates of substitution exceeds 1. To restore this equality will require increasing y. Thus the equilibrium level of y is sub-optimal.

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It must have slope $\frac{\sum_i \alpha_i}{\alpha_i}$.

If we normalize the price of the private good to be 1, this means that i must view $\frac{\alpha_i}{\sum_i \alpha_i}$ the price of the public good: personalized/Lindahl prices.

Theorem (Foley). Suppose $((\bar{x}_{\pi,i}), \bar{x}_g, \bar{y})$ is a Pareto optimal allocation. If preferences are convex, reflexive, transitive and monotonic then there exists a price vector $p = (p_\pi, p_{g,1}, \dots p_{g,m}) \in R_+^{l+km}$, (where $p_\pi \in R^l, p_{g,i} \in R^k$ for all i) such that

(a) for all i, if $x_i \succeq_i (\bar{x}_{\pi,i}, \bar{x}_g)$, then $(p_\pi, p_{g,i}) \cdot x_i \geq (p_\pi, p_{g,i}) \cdot (\bar{x}_{\pi,i}, \bar{x}_g)$.

(b) $(p_\pi, (\Sigma_{i=1}^m p_{g,i})) \cdot \bar{y} \geq (p_\pi, (\Sigma_{i=1}^m p_{g,i})) \cdot y$ for all $y \in Y$.

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Additional assumptions are needed to strengthen expenditure minimization to utility maximization in (a).

The second welfare theorem states, roughly speaking, that Pareto optimality requires that the sum of the marginal rates of substitution for a public good be equated to its marginal cost.

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Suppose there is a system of personalized market prices $p = (p_\pi, p_{g,1}, \ldots, p_{g,m})$.

The interpretation is that all consumers face the same vector of private goods prices, p_π but consumer i faces $p_{g,i}$ as the vector of public goods prices. The firm faces a price vector $(p_\pi, \Sigma_i\, p_{g,i})$.

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Consumer i's income can be defined as

$$w_i(p_i, y) = p_\pi \cdot \omega_{\pi,i} + \theta_i(p_\pi, \sum_i p_{g,i}) \cdot y.$$

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An equilibrium notion with such a price system was first proposed by Lindahl.

A **Lindahl equilibrium** is defined a vector of consumption plans (x_π, x_g), a production plan y and a vector of prices $p = (p_\pi, p_{g,1}, \ldots, p_{g,m}) \in R_+^{l+km}$ such that $p \neq 0$ and

(i) for all i, $x_i \in \gamma_i(p_i, y)$ and there does not exist $x_i' \in \gamma_i(p_i, y)$ such that $x_i' \succ_i x_i$,

(ii) $(p_\pi, \Sigma_i\, p_{g,i}) \cdot y \geq (p_\pi, \Sigma_i\, p_{g,i}) \cdot y'$ for all $y' \in Y$,

(iii) $\Sigma_i\, x_{\pi,i} = y_\pi + \Sigma_i\, \omega_{\pi,i}$, and $x_g = y_g$.

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An allocation $(x_\pi, x_g), y)$ is in the α-core if there does not exist a coalition S and an S-feasible allocation such that all members of the coalition are better-off with this new allocation assuming that the players outside S produce no public good, i.e., if there does not a coalition S and $((x'_{\pi,i})_{i\in S}, x'_g, y')$ such that

$$u_i(x'_{\pi,i}, x'_g) > u_i(x_{\pi,i}, x_g) \text{ for all } i \in S, \quad y' \in Y$$

and

$$(\sum_{i\in S} x'_{\pi,i}, x'_g) \leq \sum_{i\in S} \omega_i + y'.$$

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Proof. Suppose not. Then there exists an objection from coalition S with $((x'_{\pi,i})_{i\in S}, x'_g, y')$.

By (i) in the definition of a Lindahl equilibrium,

$$p_\pi \cdot \sum_{i\in S} x'_{\pi,i} + \sum_{i\in S} p_{g,i} \cdot x'_g > p_\pi \cdot \sum_{i\in S} \omega_{\pi,i} + \sum_{i\in S} \theta_i(p_\pi, \sum_{i\in N} p_{g,i}) \cdot y.$$

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Since $p_{g,i} \geq 0$ for all i and $(p_\pi, \sum_{i\in N} p_{g,i}) \cdot y = 0$, this means that

$$p_\pi \cdot \sum_{i\in S} x'_{\pi,i} + \sum_{i\in N} p_{g,i} \cdot x'_g > p_\pi \cdot \sum_{i\in S} \omega_{\pi,i}$$

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$$p_\pi \cdot \sum_{i\in S} x'_{\pi,i} + \sum_{i\in N} p_{g,i} \cdot x'_g > p_\pi \cdot \sum_{i\in S} \omega_{\pi,i}$$

or,

$$(p_\pi, \sum_{i\in N} p_{g,i}) \cdot y' > 0 - \text{a contradiction} \qquad \blacksquare$$

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Proof: Let $R_i = \{x_i \in X_i \mid x_i \succeq_i (\bar{x}_{\pi,i}, \bar{x}_g)\}$ for all i and let

$$R = \{(x_\pi, (x_{g,1}, \ldots, x_{g,m})) \in R^{l+km} \mid \exists x_{\pi,i},\ \Sigma_i x_{\pi,i} = x_\pi \text{ and } (x_{\pi,i}, x_{g,i}) \in R_i \text{ for all } i\}.$$

Since R_i is convex for all i, it can be shown that so is R.

Let

$\hat{Y} = \{(y_\pi, y_{g,1}, \ldots, y_{g,m}) \in R^{l+km} \mid y_{g,i} = y_g$ for all i and $y \in Y\}$.

Notice that $\hat{Y} \subseteq R^{l+km}$ and is convex.

Let $W = R - \hat{Y} - (\omega_\pi, 0, \ldots, 0)$. We shall first show that

$$0 \in Bdry(W).$$

Suppose not. Then there exists $z \ll 0$ and $z \in W$. This means that there exist $(x_{\pi,i}, x_{g,i}) \in R_i$ and $y \in Y$ such that

$$z = (\sum_i x_{\pi,i}, (x_{g,i})) - (y_\pi, y_g, \ldots, y_g) - (\omega_\pi, 0, \ldots, 0).$$

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Let $\hat{x}_{\pi,i} = x_{\pi,i} - \frac{z_\pi}{m}$, $\hat{x}_{g,i} = x_{g,i} - z_{g,i}$ so that

$$\sum_i \hat{x}_{\pi,i} = y_\pi + \omega_\pi$$

and

$$\hat{x}_{g,i} = y_g \text{ for all } i.$$

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$$\hat{x}_{g,i} = y_g \text{ for all } i.$$

From the last set of equations we can write $\hat{x}_{g,i} = \hat{x}_g$. Certainly, $((\hat{x}_{\pi,i}), \hat{x}_g, y)$ is an allocation. Since $z \ll 0$, by monotonicity and transitivity, this implies that each i prefers this allocation to the given allocation $((\bar{x}_{\pi,i}), \bar{x}_g, \bar{y})$, which contradicts the Pareto optimality of the latter.

Since W is convex and $0 \in Bdry(W)$, we can appeal to the separating hyperplane theorem to assert that there exists $p \in R^{l+km}$ such that $p \neq 0$ and

$$p \cdot w \geq 0 \text{ for all } w \in W \qquad (1).$$

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$$p \cdot w \geq 0 \text{ for all } w \in W \qquad (1).$$

To prove (a), consider $x_i \succeq_i (\bar{x}_{\pi,i}, \bar{x}_g)$ and define

$$w_i = (x_{\pi,i} + \sum_{k \neq i} \bar{x}_{\pi,k}, \bar{x}_g, \ldots x_{g,i}, \ldots, \bar{x}_g) - (\bar{y}_\pi, \bar{y}_g, \ldots, \bar{y}_g) - (\omega_\pi, 0, \ldots, 0)$$

Clearly, $w_i \in W$ and we can apply (1) to obtain,

$$p_\pi \cdot (\sum_{k \neq i} \bar{x}_{\pi,k} + x_{\pi,i} - \bar{y}_\pi - \omega_\pi)$$

$$+ \sum_{k \neq i} p_{g,k} \cdot \bar{x}_g + p_{g,i} \cdot x_{g,i} - (\sum_i p_{g,i}) \cdot \bar{y}_g \geq 0. \qquad (2)$$

Since $((\bar{x}_{\pi,i}), \bar{x}_g, \bar{y})$ is an allocation we know that

$$p_\pi \cdot (\sum_{k \neq i} \bar{x}_{\pi,k} - \bar{y}_\pi - \omega_\pi) = -p_\pi \cdot \bar{x}_{\pi,i}$$

and

$$\sum_{k \neq i} p_{g,k} \cdot \bar{x}_g - (\sum_i p_{g,i}) \cdot \bar{y}_g = -p_{g,i} \cdot \bar{x}_g.$$

Substituting this in (2) we obtain (a).

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Substituting this in (2) we obtain (a).

To establish (b) we pick any $y \in Y$ and consider

$$w = (\sum_i \bar{x}_{\pi,i}, \bar{x}_g, \ldots, \bar{x}_g) - y - (\omega_\pi, 0, \ldots, 0).$$

Clearly $w \in W$ and by the same argument as above we can obtain (b). It is easy, given monotonicity, to show that $p \geq 0$. ■