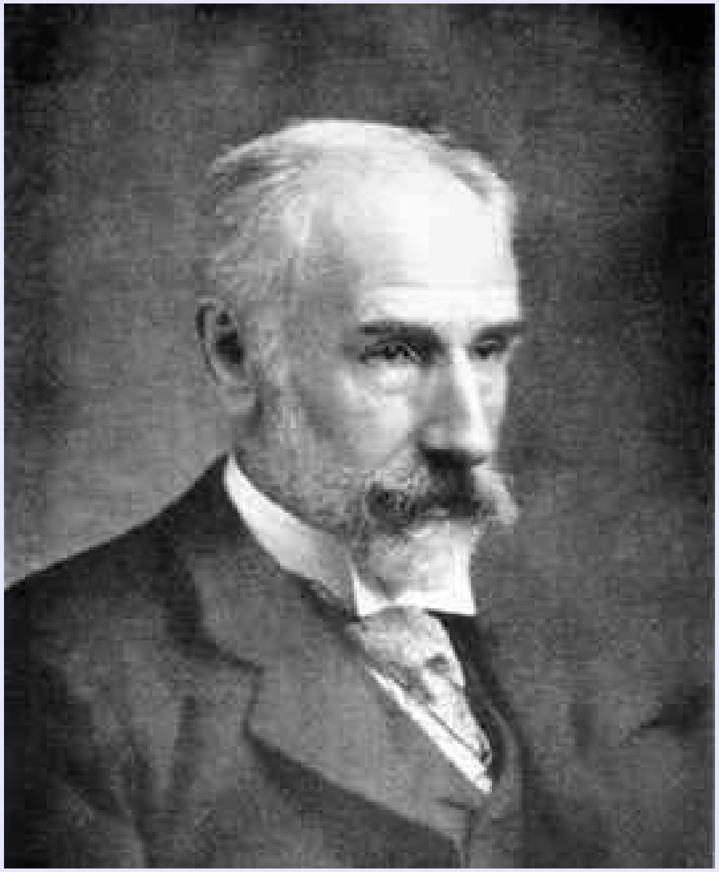
Asymmetric Information and the Core

Rajiv Vohra



Francis Ysidro Edgeworth, 1845 - 1926

- Forges, F., J.-F. Mertens and R. Vohra (2002), “The Ex Ante Incentive Compatible Core in the Absence of Wealth Effects”, *Econometrica*.

- Forges, F., E. Minelli and R. Vohra (2001), “Incentives and the Core of an Exchange Economy: A Survey”, forthcoming, *Journal of Mathematical Economics*.

- Dutta, B. and R. Vohra (2001), “Incomplete Information, Credibility and the Core”, Working Paper No. 2001-02, Department of Economics, Brown University.

Earlier papers:

- Wilson, R. (1978), "Information, Efficiency and the Core of an Economy", *Econometrica* **46**, 807–816.
- Holmström, B. and R. Myerson (1983), "Efficient and Durable Decision Rules with Incomplete Information", *Econometrica* **51**, 1799–1819.
- Vohra, R. (1999), "Incomplete Information, Incentive Compatibility and the Core", *Journal of Economic Theory* **86**, 123–147.

Two critical issues:

- Cooperation with respect to private information may be inherently infeasible – it may be important to impose incentive compatibility constraints.

- When are coalitional agreements made? At the ex ante stage (before anyone receives private information) or the interim stage (after private information is received)?

	Ex Ante Stage	Interim Stage
Without Incentive Compatibility		
With Incentive Compatibility		

Holmstrom-Myerson Taxonomy for Pareto Efficiency

	Ex Ante Stage	Interim Stage
Without Incentive Compatibility	Ex Ante Classical Efficiency	Interim Classical Efficiency
With Incentive Compatibility	Ex Ante Incentive Efficiency	(Interim) Incentive Efficiency

The Core

	Ex Ante Stage	Interim Stage
Without Incentive Compatibility	The Core of the A-D Economy with Complete Markets	Various notions, including Coarse Core; Wilson (1979)
With Incentive Compatibility	A-D economy with I.C.; Forges-Mertens-Vohra (2002)	Various notions; including I.C. Coarse Core; Vohra (1999)

The Basic Economy

Set of agents $N = \{1, \ldots, n\}$.

The private information of agent i is represented by i's type, $t_i \in T_i$, where T_i is a finite set.

$T = \prod_{i=1}^{n} T_i$ and $t = (t_i)_{i \in N}$ is a typical element of T, representing the information state.

The Basic Economy

Set of agents $N = \{1, \ldots, n\}$.

The private information of agent i is represented by i's type, $t_i \in T_i$, where T_i is a finite set.

$T = \prod_{i=1}^{n} T_i$ and $t = (t_i)_{i \in N}$ is a typical element of T, representing the information state.

Let q be a probability distribution over T. Assume $q(t_i) > 0, \forall t_i \in T_i$. However, $q(t) = 0$ for some $t \in T$ is allowed for. Let $T^* = \{t \in T | q(t) > 0\}$ denote the set of positive probability states.

Each i has an initial endowment $e_i \in \mathbf{R}^l_+$, which does not depend on his type.

$$u_i : T \times \mathbf{R}^l_+ \to \mathbf{R} \qquad i = 1, ..., n$$

such that $\forall t \in T$, $u_i(t, .)$ is increasing, continuous and concave.

The basic economy:

$$E = \{N, (T_i, u_i, e_i)_{i \in N}, q\}.$$

Each i has an initial endowment $e_i \in \mathbf{R}^l_+$, which does not depend on his type.

$$u_i : T \times \mathbf{R}^l_+ \to \mathbf{R} \qquad i = 1, ..., n$$

such that $\forall t \in T$, $u_i(t, .)$ is increasing, continuous and concave.

The basic economy:

$$E = \{N, (T_i, u_i, e_i)_{i \in N}, q\}.$$

$$X = \left\{ x = (x_i)_{i \in N} \in (\mathbf{R}^l_+)^N \mid \sum_{i \in N} x_i \leq \sum_{i \in N} e_i \right\}$$

is the set of feasible allocations.

A feasible (direct) mechanism is a function,

$$\mu : T \to X.$$

A feasible (direct) mechanism is a function,

$$\mu : T \to X.$$

Ex Ante Utility:

$$U_i(\mu) = \sum_t q(t) u_i(t, \mu_i(t))$$

A feasible (direct) mechanism is a function,

$$\mu : T \to X.$$

Ex Ante Utility:

$$U_i(\mu) = \sum_t q(t) u_i(t, \mu_i(t))$$

Interim Utility:

$$U_i(\mu \mid t_i) = \sum_{t_{-i}} q(t_{-i} \mid t_i) u_i(t_i, t_{-i}, \mu_i(t_i, t_{-i}))$$

A feasible (direct) mechanism is a function,

$$\mu : T \to X.$$

Ex Ante Utility:

$$U_i(\mu) = \sum_t q(t) u_i(t, \mu_i(t))$$

Interim Utility:

$$U_i(\mu \mid t_i) = \sum_{t_{-i}} q(t_{-i} \mid t_i) u_i(t_i, t_{-i}, \mu_i(t_i, t_{-i}))$$

If types are not verifiable, it becomes necessary to restrict attention to those mechanisms which are also informationally feasible, i.e., incentive compatible.

By reporting s_i, agent i of type t_i will get expected utility

$$U_i(\mu \mid t_i, s_i) = \sum_{t_{-i}} q(t_{-i} \mid t_i) u_i[t_i, t_{-i}, \mu_i(s_i, t_{-i})]. \tag{1}$$

By reporting s_i, agent i of type t_i will get expected utility

$$U_i(\mu \mid t_i, s_i) = \sum_{t_{-i}} q(t_{-i} \mid t_i) u_i[t_i, t_{-i}, \mu_i(s_i, t_{-i})]. \tag{1}$$

Mechanism μ is incentive compatible if and only if

$$U_i(\mu \mid t_i) \geq U_i(\mu \mid t_i, s_i) \qquad \forall i \in N, \forall t_i, s_i \in T_i. \tag{2}$$

By reporting s_i, agent i of type t_i will get expected utility

$$U_i(\mu \mid t_i, s_i) = \sum_{t_{-i}} q(t_{-i} \mid t_i) u_i[t_i, t_{-i}, \mu_i(s_i, t_{-i})]. \tag{1}$$

Mechanism μ is incentive compatible if and only if

$$U_i(\mu \mid t_i) \geq U_i(\mu \mid t_i, s_i) \qquad \forall i \in N, \forall t_i, s_i \in T_i. \tag{2}$$

By the revelation principle, for any Bayesian Nash equilibrium of a communication game, there is an equivalent incentive compatible direct mechanism.

Efficiency

Let μ and ν be feasible mechanisms;

ν ex ante dominates μ if and only if

$$U_i(\nu) > U_i(\mu) \quad \forall i \in N,$$

Efficiency

Let μ and ν be feasible mechanisms;

ν ex ante dominates μ if and only if

$$U_i(\nu) > U_i(\mu) \quad \forall i \in N,$$

A feasible mechanism μ is ex ante classically efficient if and only if there is no feasible mechanism that ex ante dominates μ.

Efficiency

Let μ and ν be feasible mechanisms;

ν interim dominates μ if and only if

$$U_i(\nu \mid t_i) > U_i(\mu \mid t_i) \qquad \forall i \in N, \forall t_i \in T_i,$$

A feasible mechanism μ is interim classically efficient if and only if there is no feasible mechanism that interim dominates μ.

Efficiency

Let μ and ν be feasible mechanisms;

ν ex post dominates μ if and only if

$$u_i(t, \nu) > u_i(t, \mu) \qquad \forall i \in N, \forall t \in T,$$

A feasible mechanism μ is ex post classically efficient if and only if there is no feasible mechanism that ex post dominates μ.

Suppose μ is incentive compatible; μ is ex ante incentive efficient if and only if there is no incentive compatible feasible mechanism that ex ante dominates μ.

Suppose μ is incentive compatible; μ is interim incentive efficient if and only if there is no incentive compatible feasible mechanism that interim dominates μ.

Ex ante efficiency implies interim efficiency, which in turn implies ex post efficiency, and this holds for both the classical and incentive notions.

Example 1 (Market for lemons).

There are two consumers and two commodities.

Suppose $T_1 = \{s, t\}$ while agent 2 is uninformed (and therefore has only one type).

The information state can then be described by s or t.

Suppose s and t are equally probable.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$$u_i(s, x^1, x^2) = x^2, \quad i = 1, 2.$$

$$u_1(t, x^1, x^2) = x^1 + x^2, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2.$$

$$u_i(s, x^1, x^2) = x^2, \quad i = 1, 2.$$

$$u_1(t, x^1, x^2) = x^1 + x^2, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2.$$

A mechanism that involves no trade in state t cannot be classically efficient.

$$u_i(s, x^1, x^2) = x^2, \quad i = 1, 2.$$

$$u_1(t, x^1, x^2) = x^1 + x^2, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2.$$

A mechanism that involves no trade in state t cannot be classically efficient.

Let z_1 denote the net trade of consumer 1. The no-trade mechanism z^*, where $z_1^*(s) = z_1^*(t) = (0, 0)$, is not interim (or ex ante) efficient.

$$u_i(s, x^1, x^2) = x^2, \quad i = 1, 2.$$

$$u_1(t, x^1, x^2) = x^1 + x^2, \qquad u_2(t, x^1, x^2) = 1.5x^1 + x^2.$$

A mechanism that involves no trade in state t cannot be classically efficient.

Let z_1 denote the net trade of consumer 1. The no-trade mechanism z^*, where $z_1^*(s) = z_1^*(t) = (0, 0)$, is not interim (or ex ante) efficient.

Consider the trade z where $z_1(s) = z_1(t) = (-1, 0.6)$. This is ex ante efficient as well as incentive compatible. Thus it is ex ante incentive efficient. It is also an ex ante improvement over z^*.

However, z is not an interim improvement over z^* because it makes the informed agent of type t worse-off.

In fact, z is interim incentive efficient.

However, z is not an interim improvement over z^* because it makes the informed agent of type t worse-off.

In fact, z is interim incentive efficient.

An interim improvement for agent 1 in state t requires a trade at which the effective price of commodity 1 is greater than the price of commodity 2. But incentive compatibility then implies the same trade in state s.

However, z is not an interim improvement over z^* because it makes the informed agent of type t worse-off.

In fact, z is interim incentive efficient.

An interim improvement for agent 1 in state t requires a trade at which the effective price of commodity 1 is greater than the price of commodity 2. But incentive compatibility then implies the same trade in state s. This results in a lower expected utility for agent 2.

However, z is not an interim improvement over z^* because it makes the informed agent of type t worse-off.

In fact, z is interim incentive efficient.

An interim improvement for agent 1 in state t requires a trade at which the effective price of commodity 1 is greater than the price of commodity 2. But incentive compatibility then implies the same trade in state s. This results in a lower expected utility for agent 2.

The fact that z does not interim dominate z^* points to the importance of making both types of the informed agent better-off at the interim stage (adverse selection).

Example 2.

The information structure is the same as in example 1. The endowments (in both states) are $e_1 = e_2 = (1, 1)$, and the utility functions are:

$$u_1(s, x^1, x^2) = x^1, \qquad u_1(t, x^1, x^2) = x^2,$$
$$u_2(s, x^1, x^2) = u_2(t, x^1, x^2) = x^1 + x^2.$$

Consider the mechanism with net-trades z, where $z_1(s) = (1, -1)$ and $z_1(t) = (-1, 1)$. This is incentive compatible as well as ex ante (and, therefore, also ex post) classically efficient. Clearly then, it is interim incentive efficient. Thus, incentive compatibility can be satisfied without sacrificing efficiency; the uninformed agent can safely delegate to the informed consumer the decision on how to trade.

The Ex Ante Core

- "Bargaining stage": a coalition may "form", in the usual sense of cooperative theory and agree upon a feasible, state contingent (possibly random) allocation.

- Nature chooses each agent's type; agent i is only informed of his own type t_i.

- If S has formed, every member i of S must report a type, and trades are executed in accordance with the announced state and the allocation rule which was agreed upon ex ante. (Strategic considerations).

A mechanism μ satisfies the physical feasibility conditions for coalition S if

$$\sum_{i \in S} \mu_i(t) \leq \sum_{i \in S} e_i \quad \forall t \in T. \tag{3}$$

Let the set of mechanisms satisfying (3) be denoted $\mathcal{F}_S$.

A mechanism μ satisfies the physical feasibility conditions for coalition S if

$$\sum_{i \in S} \mu_i(t) \leq \sum_{i \in S} e_i \quad \forall t \in T. \tag{3}$$

Let the set of mechanisms satisfying (3) be denoted $\mathcal{F}_S$.

A mechanism for S should only depend on information available within the coalition (measurability):

$$\mu_i(t) = \mu_i(t') \quad \forall i \in S, \forall t, t' \in T : t_S = t'_S. \tag{4}$$

where $t_S = (t_i)_{i \in S}$.
Let $\mathcal{F}_S^m$ denote the set of mechanisms satisfying both (3) and (4).

A mechanism $\mu \in \mathcal{F}_S^m$ is incentive compatible for S if it satisfies (2) for all $i \in S$.

Let $\mathcal{F}_S^*$ denote the set of feasible and incentive compatible mechanisms for S, i.e., $\mathcal{F}_S^*$ is the set of all mechanisms satisfying (2), (3) and (4).

A mechanism $\mu \in \mathcal{F}_S^m$ is incentive compatible for S if it satisfies (2) for all $i \in S$.
Let $\mathcal{F}_S^*$ denote the set of feasible and incentive compatible mechanisms for S, i.e., $\mathcal{F}_S^*$ is the set of all mechanisms satisfying (2), (3) and (4).

Using the notation $T \equiv T_N$, $X \equiv X_N$, etc. $\mathcal{F}_N \equiv \mathcal{F} = \mathcal{F}^m$ and $\mathcal{F}^{m*} = \mathcal{F}^*$.

Let $\mu \in \mathcal{F}^*$ and let $\nu_S \in \mathcal{F}_S^*$ for some coalition S. ν_S ex ante dominates μ for coalition S if and only if

$$U_i(\nu_S) > U_i(\mu) \quad \forall i \in S.$$

Let $\mu \in \mathcal{F}^*$ and let $\nu_S \in \mathcal{F}^*_S$ for some coalition S. ν_S ex ante dominates μ for coalition S if and only if

$$U_i(\nu_S) > U_i(\mu) \quad \forall i \in S.$$

The ex ante incentive compatible core is the set of all mechanisms $\mu \in \mathcal{F}^*$ that are not ex ante dominated by any mechanism $\nu_S \in \mathcal{F}^*_S$ for any coalition S.

Let $\mu \in \mathcal{F}^*$ and let $\nu_S \in \mathcal{F}^*_S$ for some coalition S. ν_S ex ante dominates μ for coalition S if and only if

$$U_i(\nu_S) > U_i(\mu) \quad \forall i \in S.$$

The ex ante incentive compatible core is the set of all mechanisms $\mu \in \mathcal{F}^*$ that are not ex ante dominated by any mechanism $\nu_S \in \mathcal{F}^*_S$ for any coalition S.

Observe that the set of corresponding expected payoffs is just the standard core of the game defined by the characteristic function

$$V^*(S) = \{v \in \mathbf{R}^n \mid \exists\ \mu_S \in \mathcal{F}^*_S \text{ such that } v_i \leq U_i(\mu_S)\, \forall i \in S\}.$$

One can also consider the "classical" ex ante core, which does not take account of incentive compatibility constraints. In that case, it is as reasonable to dispense with the measurability conditions (4) and consider any mechanism in $\mathcal{F}_S$.

One can also consider the "classical" ex ante core, which does not take account of incentive compatibility constraints. In that case, it is as reasonable to dispense with the measurability conditions (4) and consider any mechanism in $\mathcal{F}_S$.
The corresponding characteristic functions is:

$$V(S) = \{v \in \mathbf{R}^n \mid \exists\ \mu_S \in \mathcal{F}_S \text{ such that } v_i \leq U_i(\mu_S)\, \forall i \in S\}.$$

In the space of allocations, the corresponding core is then the core of an Arrow-Debreu economy with complete contingent markets, to which all classical (existence, convergence) results apply.

Restricting coalition S's feasible allocations to $\mathcal{F}_S^m$ just reduces the set of objections (while $\mathcal{F}^m = \mathcal{F}$), so that the associated core is still non-empty (this core corresponds to the "fine core" in Allen (1993) and to the "weak fine core" in Koutsougeras-Yannelis (1993).

Is the ex ante incentive compatible core non-empty?

Is the ex ante incentive compatible core non-empty?

Yes – in the following cases:

- In two-agent economies

Is the ex ante incentive compatible core non-empty?

Yes – in the following cases:

- In two-agent economies
- If preferences are linear; Ichiishi and Idzik (1996)

Is the ex ante incentive compatible core non-empty?

Yes – in the following cases:

- In two-agent economies
- If preferences are linear; Ichiishi and Idzik (1996)
- If information is non-exclusive; Vohra (1999)

Is the ex ante incentive compatible core non-empty?

Yes – in the following cases:

- In two-agent economies
- If preferences are linear; Ichiishi and Idzik (1996)
- If information is non-exclusive; Vohra (1999)
- If agents are informationally small; McLean and Postlewaite (1999)

Is the ex ante incentive compatible core non-empty?

Yes – in the following cases:

- In two-agent economies
- If preferences are linear; Ichiishi and Idzik (1996)
- If information is non-exclusive; Vohra (1999)
- If agents are informationally small; McLean and Postlewaite (1999)
- In assignment games; Forges (2002)

A General Approach for Non-Emptiness

The possibility of monetary transfers (quasi-linearity) can help satisfy incentive constraints; d'Aspremont and Gérard-Varet (1979, 1982) etc.

A General Approach for Non-Emptiness

The possibility of monetary transfers (quasi-linearity) can help satisfy incentive constraints; d'Aspremont and Gérard-Varet (1979, 1982) etc.

Suppose monetary transfers are available. The characteristic function is then TU. Let v denote the characteristic function without incentive constraints, and v^* the one with incentive constraints.

A General Approach for Non-Emptiness

The possibility of monetary transfers (quasi-linearity) can help satisfy incentive constraints; d'Aspremont and Gérard-Varet (1979, 1982) etc.

Suppose monetary transfers are available. The characteristic function is then TU. Let v denote the characteristic function without incentive constraints, and v^* the one with incentive constraints. Of course,

$$v^*(S) \leq v(S) \text{ for all } S \tag{5}$$

In general, (5) yields no logical inclusion between the various cores.

However, it should be clear that if

$$v^*(N) = v(N), \tag{6}$$

then

$$C(v) \subseteq C(v^*)$$

By the complete information results, $C(v)$ is not empty. Thus, it follows that the ex ante incentive compatible core is non-empty if (6) holds.

However, it should be clear that if

$$v^*(N) = v(N), \tag{6}$$

then

$$C(v) \subseteq C(v^*)$$

By the complete information results, $C(v)$ is not empty. Thus, it follows that the ex ante incentive compatible core is non-empty if (6) holds.

The literature has identified many conditions (e.g. independent private values) under which the following assumption holds:

Assumption 1: Any first-best (ex ante classically efficient) allocation can be made incentive compatible through appropriate transfers with budget balance.

When Assumption 1 holds, it can be shown that (6) also holds. Thus,

If Assumption 1 holds (e.g. independent private values), the ex ante incentive compatible core is non-empty.

When Assumption 1 holds, it can be shown that (6) also holds. Thus,

If Assumption 1 holds (e.g. independent private values), the ex ante incentive compatible core is non-empty.

To see that Assumption 1 implies (6):

Pick an allocation the core of the Arrow-Debreu economy.

Apply the transfers that yield incentive compatibility.

Another round of transfers (independent of state) to restore ex ante utilities to their original levels.

When Assumption 1 holds, it can be shown that (6) also holds. Thus,

If Assumption 1 holds (e.g. independent private values), the ex ante incentive compatible core is non-empty.

To see that Assumption 1 implies (6):

Pick an allocation the core of the Arrow-Debreu economy.

Apply the transfers that yield incentive compatibility.

Another round of transfers (independent of state) to restore ex ante utilities to their original levels.

Caution: Results relying on transfers typically rely on the availability of unlimited transfers.

Another Positive Result

This approach is also used in Forges, Mertens, Vohra (2002) for the case in which endowments vary with types. It is assumed that an agent can understate (but not overstate) his endowments. For example, agents must demonstrate to the mechanism designer the resources they are supposed to have when they claim to be of a certain type. This makes types partially verifiable. (The poor types cannot lie).

Proposition 1 *If each agent's endowment is a one-to-one function of his type, then* $v^* = v$ *and, therefore, the ex ante incentive compatible core is non-empty.*

Proof: Fix a mechanism ξ (i.e., without the transfers) for coalition S, which achieves $v(S)$. Let, for $t, a \in T^i$, u^i_{ta} denote i's expected utility from ξ when of type t and claiming to be of type a, i.e., $u^i_{ta} = U_i((\xi, 0)|t, a)$.

Proof: Fix a mechanism ξ (i.e., without the transfers) for coalition S, which achieves $v(S)$. Let, for $t, a \in T^i$, u^i_{ta} denote i's expected utility from ξ when of type t and claiming to be of type a, i.e., $u^i_{ta} = U_i((\xi, 0)|t, a)$.
Consider $\mu^i \in R^{T^i}$ such that, whenever $e^i_t \geq e^i_a$ (i.e., a is available to t), $\mu^i_t - \mu^i_a \geq u^i_{ta} - u^i_{tt}$:

Proof: Fix a mechanism ξ (i.e., without the transfers) for coalition S, which achieves $v(S)$. Let, for $t, a \in T^i$, u^i_{ta} denote i's expected utility from ξ when of type t and claiming to be of type a, i.e., $u^i_{ta} = U_i((\xi, 0)|t, a)$.
Consider $\mu^i \in R^{T^i}$ such that, whenever $e^i_t \geq e^i_a$ (i.e., a is available to t), $\mu^i_t - \mu^i_a \geq u^i_{ta} - u^i_{tt}$:
there exist such μ^i's, by the one-to-one assumption: e.g., with $K = \max_{s,t}(u^i_{ts} - u^i_{tt})$, let $\mu^i_t = K \cdot \#\{\, s \in T^i \mid e^i_s \leq e^i_t \,\}$.

Proof: Fix a mechanism ξ (i.e., without the transfers) for coalition S, which achieves $v(S)$. Let, for $t, a \in T^i$, u^i_{ta} denote i's expected utility from ξ when of type t and claiming to be of type a, i.e., $u^i_{ta} = U_i((\xi, 0)|t, a)$.
Consider $\mu^i \in R^{T^i}$ such that, whenever $e^i_t \geq e^i_a$ (i.e., a is available to t), $\mu^i_t - \mu^i_a \geq u^i_{ta} - u^i_{tt}$:
there exist such μ^i's, by the one-to-one assumption: e.g., with $K = \max_{s,t}(u^i_{ts} - u^i_{tt})$, let $\mu^i_t = K \cdot \#\{\, s \in T^i \mid e^i_s \leq e^i_t \,\}$.
ξ together with transfers μ^i_t to agent i claiming to be of type t is incentive compatible.

Proof: Fix a mechanism ξ (i.e., without the transfers) for coalition S, which achieves $v(S)$. Let, for $t, a \in T^i$, u^i_{ta} denote i's expected utility from ξ when of type t and claiming to be of type a, i.e., $u^i_{ta} = U_i((\xi, 0)|t, a)$.
Consider $\mu^i \in R^{T^i}$ such that, whenever $e^i_t \geq e^i_a$ (i.e., a is available to t), $\mu^i_t - \mu^i_a \geq u^i_{ta} - u^i_{tt}$:
there exist such μ^i's, by the one-to-one assumption: e.g., with $K = \max_{s,t}(u^i_{ts} - u^i_{tt})$, let $\mu^i_t = K \cdot \#\{\, s \in T^i \mid e^i_s \leq e^i_t \,\}$.
ξ together with transfers μ^i_t to agent i claiming to be of type t is incentive compatible.

Whenever (ξ, μ) is incentive compatible, and, $\forall i \in N, \forall t \in T_S$, $\mu^i(t)$ depends only on t^i, then

$$\nu^i(t) = \mu^i(t^i) - \frac{1}{\#S - 1} \sum_{j \in S \setminus \{i\}} \mu^j(t^j)$$

is such that (ξ, ν) is incentive compatible and budget-balanced.■

But non-emptiness does depend on such additional assumptions.

Example 2 (Forges-Mertens-Vohra (2002)).

Three agents;

Four goods (three consumption goods and money);

Agent 1 has two equiprobable types s and t, while agents 2 and 3 do not have private information:

$(T_1 = \{s, t\}, \; q(s) = q(t) = \frac{1}{2})$.

$$e_1 = (1,0,0), \quad e_2 = (0,2,0), \quad e_3 = (0,0,2).$$

$$u_i(r,x,m) = w_i(r,x) + m \quad i = 1,2,3,\ r = s,t,$$

$$w_1(s,x) = 2x^1 + \min\{(x^2 + x^3), 2\},$$
$$w_2(s,x) = w_3(s,x) = 3x^1 + 2\min\{x^2, x^3\},$$

$$w_1(t,x) = x^1 + h(x^2) + h(x^3),$$
$$w_2(t,x) = w_3(t,x) = x^2 + x^3,$$

h is concave, with $h(0) = 0$, $h(x) \leq x$, $h \leq 1$ and $h(1) > .5$.

It is important that

- The endowments are not one-to-one with the states
- The utilities of the uninformed agents depend on information they do not possess (non-private values)
- Utility functions are not separable in commodities.

Otherwise, the ex ante core would be non-empty.

A TU game has a non-empty core if and only if it is balanced.

In a superadditive, three-player, TU game balancedness reduces to the following condition:

$$v(N) \geq \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

A TU game has a non-empty core if and only if it is balanced.

In a superadditive, three-player, TU game balancedness reduces to the following condition:

$$v(N) \geq \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

A coalitional game derived from a convex exchange economy (TU or not) is necessarily balanced, and therefore, by Scarf's theorem, has a non-empty core.

A TU game has a non-empty core if and only if it is balanced.

In a superadditive, three-player, TU game balancedness reduces to the following condition:

$$v(N) \geq \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

A coalitional game derived from a convex exchange economy (TU or not) is necessarily balanced, and therefore, by Scarf's theorem, has a non-empty core.

Thus, in the absence of incentive constraints the ex ante core in the present example is non-empty.

But $C(v^*)$ is empty.

But $C(v^*)$ is empty.

The proof goes through three steps.

But $C(v^*)$ is empty.

The proof goes through three steps.

Step 1: v is exactly balanced, i.e.

$$v(N) = \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

But $C(v^*)$ is empty.

The proof goes through three steps.

Step 1: v is exactly balanced, i.e.

$$v(N) = \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

Step 2: In subcoalitions, incentives entail no efficiency-loss:

$$v^*(S) = v(S) \quad \text{for every } S\colon |S| = 2$$

But $C(v^*)$ is empty.

The proof goes through three steps.

Step 1: v is exactly balanced, i.e.

$$v(N) = \frac{1}{2}[v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})]$$

Step 2: In subcoalitions, incentives entail no efficiency-loss:

$$v^*(S) = v(S) \quad \text{for every } S\colon |S| = 2$$

Step 3: $v^*(N) < v(N)$.

These 3 steps show that v^* is not balanced. Thus $C(v^*)$ is empty.

Step 1: v is exactly balanced.

$$w_1(s,x) = 2x^1 + \min\{(x^2 + x^3), 2\},$$
$$w_2(s,x) = w_3(s,x) = 3x^1 + 2\min\{x^2, x^3\},$$

$$w_1(t,x) = x^1 + h(x^2) + h(x^3),$$
$$w_2(t,x) = w_3(t,x) = x^2 + x^3.$$

Implications of classical efficiency:

In state s, efficiency in the grand coalition requires, $x_1^1 = 0$, $x_2^1 + x_3^1 \leq 2$, $x_2^i = x_3^i \ \forall i$, leading to a total utility of 7.

In state t, efficiency implies no-trade with agent 1, and total utility is 5.

$$v(N) = 6.$$

$$w_1(s,x) = 2x^1 + \min\{(x^2 + x^3), 2\},$$
$$w_2(s,x) = w_3(s,x) = 3x^1 + 2\min\{x^2, x^3\},$$

$$w_1(t,x) = x^1 + h(x^2) + h(x^3),$$
$$w_2(t,x) = w_3(t,x) = x^2 + x^3.$$

Coalition $\{1,2\}$ (or $\{1,3\}$): In state s efficiency requires swapping their endowments - aggregate utility is 5. And no trade in state t, aggregate utility is 3. Thus:

$$v(\{1,2\}) = v(\{1,3\}) = 4.$$

$$w_1(s,x) = 2x^1 + \min\{(x^2 + x^3), 2\},$$
$$w_2(s,x) = w_3(s,x) = 3x^1 + 2\min\{x^2, x^3\},$$

$$w_1(t,x) = x^1 + h(x^2) + h(x^3),$$
$$w_2(t,x) = w_3(t,x) = x^2 + x^3.$$

For coalition $\{2,3\}$ their commodities are perfect complements in state s. In state t all allocations are efficient. By trading to $(0,1,1),(0,1,1)$ aggregate utility is 4 on each state, and

$$v(\{2,3\}) = 4.$$

We have shown that

$$v(S) = 4 \text{ for all } S,\ |S| = 2$$

and

$$v(N) = 6.$$

Thus v is exactly balanced.

Step 2: $v^*(S) = v(S)$ for all S, $|S| = 2$

Coalition $\{2, 3\}$ faces no incentive constraints, so

$$v^*(\{2, 3\}) = v(\{2, 3\}).$$

Step 2: $v^*(S) = v(S)$ for all S, $|S| = 2$

Coalition $\{2, 3\}$ faces no incentive constraints, so

$$v^*(\{2, 3\}) = v(\{2, 3\}).$$

Now consider coalition $\{1, 2\}$.

Classical efficiency implies swapping endowments in state s and no-trade in state t.

Step 2: $v^*(S) = v(S)$ for all S, $|S| = 2$

Coalition $\{2, 3\}$ faces no incentive constraints, so

$$v^*(\{2, 3\}) = v(\{2, 3\}).$$

Now consider coalition $\{1, 2\}$.

Classical efficiency implies swapping endowments in state s and no-trade in state t.

This entails no incentive problems.

Step 3: $v^*(N) < v(N)$, i.e., a classically efficient allocation cannot satisfy incentive constraints.

By step 1, classical efficiency requires that he consumption of agent 1 is of the following form:

In state s: $(0, x, x)$ where $0 \leq x \leq 1$,

in state t: $(1, 0, 0)$.

Step 3: $v^*(N) < v(N)$, i.e., a classically efficient allocation cannot satisfy incentive constraints.

By step 1, classical efficiency requires that he consumption of agent 1 is of the following form:

In state s: $(0, x, x)$ where $0 \leq x \leq 1$,

in state t: $(1, 0, 0)$.

Using $u_s(\cdot)$ for the utility of player 1 of type s as a function of his announcement "$\cdot$", and similarly for u_t:

$$u_s(s) = 2x\,; u_s(t) = 2\,; u_t(s) = 2h(x)\,; u_t(t) = 1$$

Step 3: $v^*(N) < v(N)$, i.e., a classically efficient allocation cannot satisfy incentive constraints.

By step 1, classical efficiency requires that he consumption of agent 1 is of the following form:

In state s: $(0, x, x)$ where $0 \leq x \leq 1$,

in state t: $(1, 0, 0)$.

Using $u_s(\cdot)$ for the utility of player 1 of type s as a function of his announcement "$\cdot$", and similarly for u_t:

$$u_s(s) = 2x\,; u_s(t) = 2\,; u_t(s) = 2h(x)\,; u_t(t) = 1$$

The incentive constraint is:

$$2x + 1 \geq 2 + 2h(x)$$

or

$$1 + 2h(x) - 2x \leq 0.$$

or

$$1 + 2h(x) - 2x \leq 0.$$

But this is impossible because

$1 + 2h(x) - 2x$ is a concave function

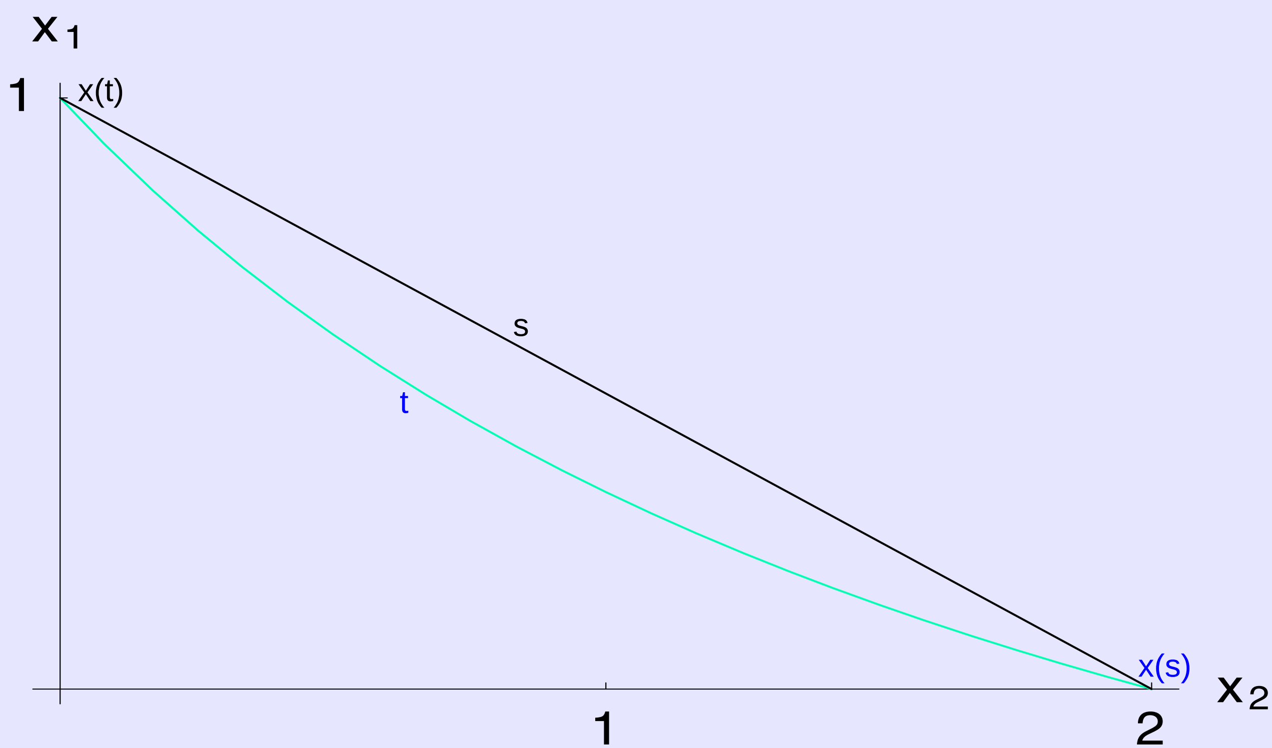
or

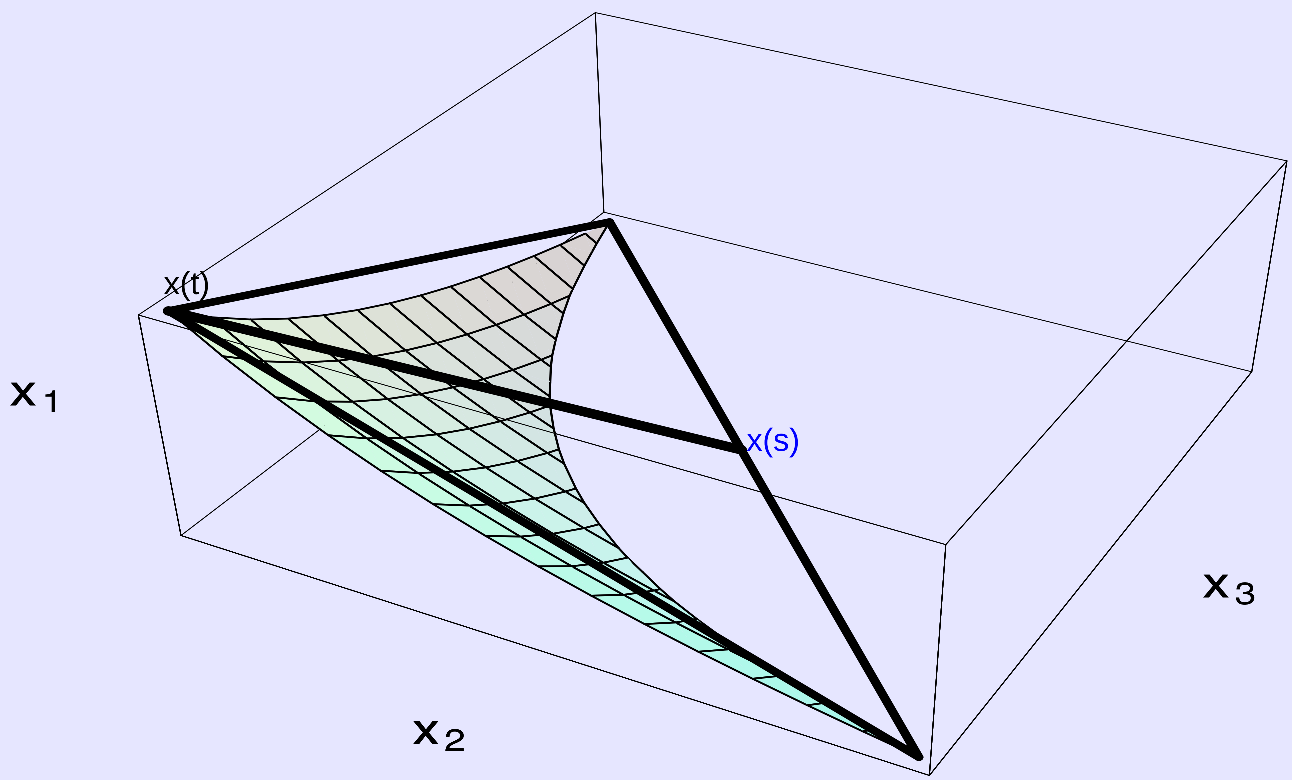
$$1 + 2h(x) - 2x \leq 0.$$

But this is impossible because

$1 + 2h(x) - 2x$ is a concave function

and is positive both at 0 and 1.


x 1
1
x(t)
s
t
x(s)
x 2
1
2


x(t)
x₁
x(s)
x₃
x₂

All the previous arguments can be extended to the case in which coalitions are permitted to use random mechanisms.

All the previous arguments can be extended to the case in which coalitions are permitted to use random mechanisms.

Why doesn't randomization help?

All the previous arguments can be extended to the case in which coalitions are permitted to use random mechanisms.

Why doesn't randomization help?

Let $x_{S,i}$ denote the state s allocations of the goods to agent i in coalition $S \in \mathcal{S}$ corresponding to the deterministic, first best efficient mechanism of S.

$$\begin{aligned} x_{\{1,2\},1} &= (0,2,0), & x_{\{1,2\},2} &= (1,0,0), \\ x_{\{1,3\},1} &= (0,0,2), & x_{\{1,3\},3} &= (1,0,0), \\ x_{\{2,3\},2} &= (0,1,1), & x_{\{2,3\},3} &= (0,1,1). \end{aligned}$$

All the previous arguments can be extended to the case in which coalitions are permitted to use random mechanisms.

Why doesn't randomization help?

Let $x_{S,i}$ denote the state s allocations of the goods to agent i in coalition $S \in \mathcal{S}$ corresponding to the deterministic, first best efficient mechanism of S.

$$\begin{aligned} x_{\{1,2\},1} &= (0,2,0), & x_{\{1,2\},2} &= (1,0,0), \\ x_{\{1,3\},1} &= (0,0,2), & x_{\{1,3\},3} &= (1,0,0), \\ x_{\{2,3\},2} &= (0,1,1), & x_{\{2,3\},3} &= (0,1,1). \end{aligned}$$

Can we find a random mechanism (feasible in each state) such that it allocates to agent i, in state s, either $x_{\{i,j\},i}$ or $x_{\{i,k\},i}$ with equal probability?

Agent 1 is supposed to receive (0,2,0) or (0,0,2) with probability .5 each. And agent 2 is supposed to get (0,1,1) with probability .5. But this is not possible.

Agent 1 is supposed to receive (0,2,0) or (0,0,2) with probability .5 each. And agent 2 is supposed to get (0,1,1) with probability .5. But this is not possible.

The counter-example is also fully robust with respect to the probabilities on states, the endowments, and the utility functions.