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Incomplete Information, Incentive Compatibility, and the Core*

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We consider an exchange economy in which agents possess private information at the time of engaging in state contingent contracts. While communication of private information is permitted, the true information state is not verifiable. The enforcement of contracts is, therefore, limited by incentive compatibility constraints. We formalize a notion of the core for such an economy. Our analysis can be viewed as an attempt to incorporate incentive compatibility in the coarse core of R. Wilson (1978, *Econometrica* **46**, 807–816), or as an attempt to introduce coalitional contracts in the notion of incentive efficiency of B. Holmstrom and R. Myerson (1983, *Econometrica* **51**, 1799–1819). While there are some special cases in which the incentive compatible core is non-empty, our main result shows that is not generally true. *Journal of Economic Literature* Classification Numbers: C71, C72, D51.

1. INTRODUCTION

The aim of this paper is to analyze the core of an economy with incomplete information. We consider an economy in which agents possess private information at the time of making a contract and engage in state contingent contracts or net-trades.

Given the presence of incomplete information, it is important to be clear about the nature of contracts over which cooperation is possible. In the

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spirit of cooperative analysis, we shall assume that the only possible restrictions on enforcing a contract (that satisfies the physical feasibility constraints) are those that stem from the presence of incomplete information. To emphasize this, we assume that there is an enforcement agency that can enforce contracts.[1] In particular, if all information becomes publicly known, and verifiable, at the time of enforcing contracts (the case studied in the seminal paper of Wilson [38] on the subject), then any state contingent contract is enforceable. In contrast, we shall assume, throughout this paper, that private information is inherently private, and is not verifiable by the enforcement agency. This means that cooperation (or the writing of binding agreements) with respect to private information is not feasible. Of course, it would still be possible to enforce a contract that is independent of the information state, and, therefore trivial with respect to state contingencies.

The fact that cooperation with respect to private information is infeasible does not necessarily imply that *only* trivial state contingent contracts are feasible. A richer set of contracts can be allowed for if agents are permitted to communicate their private information. Suppose a group of agents agree upon a state contingent contract and communicate their private information to the enforcement agency, which uses this information to enforce the contract. Since the enforcement agency only has access to the information conveyed by the agents themselves (and this information is not verifiable), it cannot rule out the possibility of misrepresentation of private information. However, any contract that relies on communication which is self-enforcing can be implemented by such a mechanism. As is well known, (see, for example, Myerson [31] and Holmstrom and Myerson [20]) the set of all such contracts is simply the set of *incentive compatible contracts*.[2] By restricting attention to incentive compatible contracts, therefore, it becomes possible for agents to engage in such state contingent contracts even though they individually, or the enforcement agency, may not know the true information state. We will take the set of incentive compatible contracts to be the feasible set of contracts over which coalitions can make binding agreements.[3]

It will be possible to view our analysis of the core as an attempt to either incorporate incentive compatibility in Wilson's framework [38], or to introduce coalitional contracts (in addition to those for the grand coalition) in the Holmstrom–Myerson [20] notion of incentive efficiency.

[1] As is implicit in the standard analysis of the core an exchange economy with perfect information. This also implies that physical actions are observable, i.e., issues of moral hazard are assumed away.

[2] The formal definition of incentive compatibility appears in the following Section.

[3] For example, the grand coalition can, in principle, choose any incentive efficient contract.

It is important in our framework to distinguish between *communication* and *information sharing*. Communication through a mediator simply allows agents to consider a rich set of potential contracts. They can consider executing contracts that are contingent on information that they may not individually possess at the interim stage—the stage at which each agent knows only her own private information. For example, in the well-known lemons problem (Akerlof [1]), it is possible for the agents to consider an allocation that varies with the information state (the quality of the object). To be sure, incentive compatibility does restrict the set of state contingent allocations that are feasible, but it does not necessarily rule out all allocations that vary with the state. Thus, the information structure, through the incentive compatibility constraints, has an important bearing on the set of contracts that communication makes feasible. But the possibility of communication by itself does not have any implications on whether or not information is *shared*.

The information available to a coalition determines the set of events[4] that the coalition can discern. And the set of events that are discernible by an agent will certainly affect her evaluation of contracts. To use the lemons problem as an illustration, the informed seller of the object can discern the actual state. For the uninformed buyer, the only discernible event is the coarse event consisting of all possible states. In evaluating a contract therefore, the seller will consider her utility in the true state while the buyer will consider her expected utility (over all the states). The less information a coalition has, the coarser will be the set of events it can discern, and the greater will be the opportunities for insurance. Notice that if contracts are executed and, therefore, evaluated at the ex-ante stage, the issue of which events are discernible by an agent, or by a coalition, is easily resolved. The coarse event consisting of all possible states is the only (non-empty) event that is discernible by any agent. And all agents then evaluate contracts in terms of expected utility. But this simplification accorded by the ex-ante framework[5] comes at the cost of ignoring many of the interesting issues relating to adverse selection. The main interest of the present paper lies in the interim framework, and is justified by the importance of adverse selection problems, as exemplified by the lemons model.

We shall consider a model in which information is not shared (or cannot be shared) by members of a coalition. This implies (as in Wilson's notion of the coarse core) that the only events discernible by a coalition are those that are commonly known, at the interim stage, to the agents in that coalition.

[4] An event is a subset of the set of information states.

[5] We do consider the ex-ante framework in Section 4. There, and in Section 5, we also relate our results to the recent work of Allen [2, 3, 5], Ichiishi and Idzik [21], Koutsougeras and Yannelis [26], and Hahn and Yannelis [16, 17].

A coalition will be permitted to object over an event if and only if that event is common knowledge to all members of the coalition. In general, the larger the coalition, the coarser will be its set of discernible (or common knowledge) events. For example, a singleton coalition can object over an event that is consistent with its own private information while the grand coalition must focus its objections only those events that are commonly known to all the agents.

The idea that a coalition should be allowed to focus its objections on an event that is commonly known to all its members is easily justified. On the other hand, the restriction that a coalition is not permitted to object on an event that is not commonly known is a real one. Even if information is inherently private, there may exist credible ways for a coalition to coordinate on an event that is not necessarily commonly known a priori. This is a complex issue, studied in various contexts by Crawford [9], Dutta and Vohra [10], Ichiishi and Sertel [23], Holmstrom and Myerson [20] and Volij [37],[6] but one that is beyond the scope of the present paper. Such considerations, by making it easier for coalitions to object, may well serve to refine the incentive compatible core that we study in this paper. Our main result, however, is robust to such refinements since it concerns the possibility that the core may be empty. We show that while there are some special cases in which the incentive compatible core is non-empty, this is not generally true. In other words, Wilson's result [38] on the non-emptiness of the coarse core does not survive the imposition of incentive compatibility.

2. THE FRAMEWORK

In specifying the private information of agents we shall find it convenient (especially in dealing with issues of incentive compatibility) to adopt Harsanyi's approach [19] of identifying the information available to agent i by i's *types*. Let T_i denote the (finite) set of agent i's types. The interpretation is that $t_i \in T_i$ denotes the *private information* possessed by agent i. With $N = \{1, ..., n\}$ as the finite set of agents, let $T = \prod_{i \in N} T_i$. An information *state* for the economy refers to $t \in T$.[7] We will use the notation t_{-i} to denote $(t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$. Similarly $T_{-i} = \prod_{j \neq i} T_j$, and for any coalition S, a non-empty subset of N, $t_S = (t_i)_{i \in S}$ and $T_S = \prod_{i \in S} T_i$. Each agent i has a prior probability distribution q_i defined on T. We assume that none of the types is redundant, in the sense that for every $i \in N$ and $t_i \in T_i$, there

[6] In a somewhat different vein, Forges [11, 12] investigates the posibility of renegotiation arising from updated conditional probabilities following a particular outcome.

[7] While aggregate uncertainty is implicitly ruled out by this approach, it should be clear that it can easily be incorporated.

exists $t_{-i} \in T_{-i}$ such that $q_i(t) > 0$. For each $i \in N$ and $\bar{t}_i \in T_i$, the conditional probability of $t_{-i} \in T_{-i}$, given $\bar{t}_i$, is

$$q_i(t_{-i} \mid \bar{t}_i) = \frac{q_i(t_{-i}, \bar{t}_i)}{\sum_{t'_{-i} \in T_{-i}} q_i(t'_i, \bar{t}_i)}.$$

It is also assumed that if $q_i(t) > 0$ for some $i \in N$ and $t \in T$, then $q_j(t) > 0$ for all $j \in N$. Let $T^* = \{t \in T \mid q_i(t) > 0, \text{ for all } i \in N\}$. Agent i of type $\bar{t}_i \in T_i$ observes the event $P_i(\bar{t}_i) = \{t \in T^* \mid t_i = \bar{t}_i\}$. In this way we can define for each i a partition $P_i = \{P_i(t_i)\}_{t_i \in T_i}$ over T^* which represents the events that are discernible by i. While some authors use partitions over the state space as the primitive formulation of private information, the two approaches are equivalent.[8]

We assume that there are a finite number of commodities, so that the consumption set of agent i in state t, denoted $X_i(t)$, is a subset of $\mathbb{R}^l_+$. The characteristics of an agent, namely, the consumption set, the endowments and utility function will depend in general on the state. We assume, for simplicity, that $X_i(t) = \mathbb{R}^l_+$ for all t, and the endowment of agent i is defined by a function ω_i: $T_i \mapsto \mathbb{R}^l_+$. Thus $\omega_i(t_i) \in \mathbb{R}^l_+$ denotes i's endowment when her type is t_i. Each consumer has a state dependent utility function u_i: $\mathbb{R}^l_+ \times T \mapsto \mathbb{R}$. We will denote by $u_i(\cdot, t)$ the von Neumann–Morgenstern utility function of agent i in state t. We assume that for each $i \in N$ and $t \in T$, $u_i(\cdot, t)$ is continuous and concave. We can now define an exchange economy as $\mathscr{E} = \{(u_i, X_i, \omega_i, T_i, q_i)_{i \in N}\}$.

Our main interest is in studying exchange that is contracted upon during the interim stage, i.e., after each agent has learnt her type. A contract for coalition S consists of a state contingent net-trade that is feasible given the endowments of coalition S in every state. In order to allow for a rich set of feasible contracts, we assume that agents can communicate freely. In many cases, allowing for self-enforcing communication of information significantly enlarges the set of contracts that are enforceable. More generally, as in Holmstrom and Myerson [20], think of the consumers as agreeing upon a mechanism to implement a state contingent contract. We shall adopt the view that a contract is enforceable if and only if it can be sustained as a Bayesian Nash equilibrium of a corresponding communication game. By the revelation principle (see, for example, Chapter 6 of Myerson [32]), we can restrict ourselves, without loss of generality, to direct mechanisms. Alternatively, the set of enforceable contracts is simply the set of *incentive compatible* contracts. This effectively defines the feasible set of state contingent contracts over which cooperative agreements can be made.

[8] See footnotes 3 and 4 in Jackson [24].

We turn now to the formal definitions.

A state contingent *contract* is a function $x: T \mapsto \prod_i X_i$, describing the consumption plans in each information state. A *feasible* contract must satisfy the resource constraints in each state. Thus, the set of contracts feasible for the grand coalition is defined as

$$A_N = \left\{ x: T \mapsto \prod_i X_i \,\middle|\, \sum_{i \in N} x_i(t) = \sum_{i \in N} \omega_i(t), \text{ for all } t \in T \right\}.$$

A contract x is *feasible for coalition* S if

(i) $\sum_{i \in S} x_i(t) = \sum_{i \in S} \omega_i(t)$ for all $t \in T$,

(ii) $x_i(t) = x_i(t')$ for all $i \in S$, t and t' such that $t_S = t'_S$.

The set of feasible contracts for coalition S is denoted A_S.

Condition (ii) requires a feasible contract for coalition S to depend only on t_S, i.e., it must be measurable with respect to the information available to S. This requirement, that a contract for a coalition can be contingent only on the information possessed by the members of the coalition, is in keeping with our interpretation of a contract as a mechanism; see Remark 3.3 for further discussion. It is only through condition (ii) that the information available to a coalition enters into the definition of a feasible contract. (Recall that t denotes the information state, not the information available to a coalition).

The conditional expected utility of consumer i corresponding to contract x, conditional on her being of type t_i, is

$$U_i(x_i \mid t_i) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} \mid t_i)\, u_i(x_i(t), t).$$

By pretending to be of type s_i, when the true type is t_i, agent i can obtain the net-trade corresponding to the state (t_{-i}, s_i) when the true state is t. Let $z(x)$ denote the net-trade corresponding to contract x, i.e., $z_i(x, t) = x_i(t) - \omega_i(t)$. This deception yields conditional expected utility

$$U_i(x_i, s_i \mid t_i) = \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} \mid t_i)\, u_i(z_i(x, t_{-i}, s_i)) + \omega_i(t_i)), t),$$

where we use the convention that $u_i(y, t) = -\infty$ for all $y \notin \mathbb{R}^l_+$ for all $i \in N$ and $t \in T$.

The set of *Bayesian incentive compatible contracts* for coalition S is

$$A^*_S = \{(x_i) \in A_S \mid U_i(x_i \mid t_i) \geqslant U_i(x_i, s_i \mid t_i) \text{ for all } s_i, t_i \in T_i, \text{ for all } i \in S\}.$$

This definition of incentive compatibility is a standard one, and similar notions of incentive compatibility have already been introduced in the literature on the core in the presence of differential information; see, for example, Allen [2, 4], Berliant [6], Boyd and Prescott [7], and Marimon [28].

Since we are examining a setting in which agents cannot pool their information, we shall follow Wilson's notion [38] of the coarse core and assume that an objecting coalition must ensure that its members can gain in an event that is discernible by all of them given their private information. This is equivalent[9] to assuming that it is common knowledge within an objecting coalition that all its members can do better (in terms of conditional expected utility) than the status-quo. The following definition, from Holmstrom and Myerson [20], is appropriate for our framework.

An event $E=\prod_{i\in N} E_i$, where $E_i\subseteq T_i$ for all i is said to be *discernible* by coalition S (or *common knowledge* within S) if $q_i(\hat{t}_{-i}\mid t_i)=0$ for all $i\in S$, $t\in E$ and $\hat{t}\notin E$.

It is only for notational convenience that we have chosen to define a common knowledge event as a cartesian product of subsets of T_i; specifying such an event as a subset of T does not alter anything of substance. Note also that by the assumptions we have already made, if E is discernible by S, then for every $i\in S$ and $t_i\in E_i$, $\sum_{t'_{-i}\in E_{-i}} q_i(t'_{-i}, t_i)>0$; the event E is assigned a positive probability by all agents in S.

Coalition S has an *objection* to $x\in A^*_N$ if there exists $y\in A^*_S$ and an event E that is discernible by S such that

$$U_i(y_i\mid t_i)>U_i(x_i\mid t_i) \qquad \text{for all} \quad t_i\in E_i \quad \text{for all} \quad i\in S. \tag{1}$$

The *incentive compatible core* consists of all contracts $x\in A^*_N$ to which there exists no objection.[10] Alternatively, an incentive compatible contract x belongs to the incentive compatible core if there does not exist a coalition S, an incentive compatible contract y and an event $E\subseteq T$ such that

(i) E is discernible by S, i.e., $q_i(\hat{t}_{-i}\mid t_i)$ for all $i\in S$, $t\in E$ and $\hat{t}\notin E$,

(ii) $U_i(y_i\mid t_i)>U_i(x_i\mid t_i)$ for all $t_i\in E_i$, for all $i\in S$,

(iii) $\sum_{i\in S} y_i(t)=\sum_{i\in S}\omega_i(t)$ for all $t\in T$,

(iv) $x_i(t)=x_i(t')$ for all $i\in S$, t and t' such that $t_S=t'_S$.

An allocation $x\in A^*_N$ is *incentive efficient* if it has no objection from the grand coalition.

[9] See especially Milgrom [29]. This point is also discused in Kobayashi [25].

[10] This notion of the core may be more accurately described as Wilson's coarse core with incentive compatibility applied to feasible contracts.

According to our definition of an objection, it is enough for the objecting coalition to be able to improve upon the status-quo over a discernible event. For the grand coalition, this turns out to be equivalent to the requirement that the new allocation dominate the status-quo for all consumers of *all* types. The argument follows from Theorem 1 of Holmstrom and Myerson [20]. In other words, for $S=N$, $y \in A_N^*$ is an objection to $x \in A_N^*$ if

$$U_i(y_i \mid t_i) \geqslant U_i(x_i \mid t_i) \quad \text{for all} \quad t_i \in T_i \quad \text{for all} \quad i \in N,$$

$$\text{with strict inequality for some} \quad t_i \in T_i \quad \text{for all} \quad i \in N.$$

This observation shows that our definition of incentive efficiency is identical[11] to that in Holmstrom and Myerson [20]. In the context of an exchange economy, it is important to realize, however, that a similar result does not generally hold for objections from coalitions other than the grand coalition. This is so because a coalition which is a strict subset of N may have an objection over a common knowledge event but may not be able to assure itself of the status-quo utility over states not in E. The reason that this is so (and correctly so) can be illustrated by considering the notion of (interim) individual rationality. Agent i knows her own type, i.e., for every $t_i' \in T_i$, the event $\{t \in T \mid t_i = t_i'\}$ is common knowledge to coalition $\{i\}$. Thus our definition of an objection would allow coalition $\{i\}$ to object to $x \in A_N^*$ if

$$U_i(\omega_i \mid t_i') > U_i(x_i \mid t_i') \quad \text{for } some \quad t_i' \in T_i,$$

without implying that *all* types of agent i improve upon the status-quo. Correspondingly, $x \in A_N^*$ is *interim individually rational* if

$$U_i(x_i \mid t_i) \geqslant U_i(\omega_i \mid t_i) \quad \text{for all} \quad i \in N, \quad t_i \in T_i,$$

which is precisely the standard notion of interim individual rationality; see Example 2.1 for an illustration.

Our definitions formalize the idea that a coalition should be allowed to coordinate its actions on any common knowledge event, and conversely that it cannot coordinate its actions on an event that is not commonly discernible. This is fairly standard (and goes back at least to Wilson [38] and Holmstrom and Myerson [20]) in a model where information cannot be pooled. However, some of the recent literature on the core takes a significantly different approach to this issue. The following, simple example, is meant to illustrate these differences, and to highlight the consequences

[11] Except for the fact that, to be consistent with the usual definition of the core, we require all inequalities to be strict.

of departing from the assumption that coalitions may coordinate their actions on an event if and only if it is a common knowledge event within the coalition.

EXAMPLE 2.1. Consider a simple version of Akerlof's [1] model of a market for lemons. Agent 1 is a seller of one unit of an indivisible object. The information state refers to the quality of this object, low or high. Agent 1 knows the true state while agent 2 places equal probabilities on the two states. Let $T=\{l, h\}$, where $P_1=(\{l\}, \{h\})$ and $P_2=\{l, h\}$. Let commodity 1 be the indivisible object and let commodity 2 refer to money. We use $x_{ih}(t)$ to denote consumer i's consumption of commodity h in state t. The preferences of the two agents are defined by the value (in terms of commodity 2) that they assign to the object in the two states. Let $v_i(t)$ be the value of agent i in state t, and suppose

$$v_1(l)=0, \quad v_1(h)=10, \qquad v_2(l)=0, \quad v_2(h)=15$$

Individual rationality for agent 1 (the informed agent) implies that if the object is traded in the high state then its price in that state must be at least 10. Incentive compatibility immediately implies that it is also sold in the low state at the same price. However, any such contract would violate individual rationality for agent 2. Thus the only individually rational and incentive compatible contracts are those which involve no trade in the high state, and no exchange of commodity 2 in either state. Let Z^* be the set of all such contracts. Clearly, the incentive compatible core is Z^*.

Note that consumer 2 would be better off with the (incentive compatible) contract z, where $z_2(l)=z_2(h)=(1, -6)$, since she is completely uninformed and this provides here a higher expected utility than her endowment. But the contract z is not individually rational for consumer 1 of type h. This illustrates the familiar problem of adverse selection.[12] Consumer 1 would agree to this only if the state is l, in which case it would be detrimental to consumer 2. A no-trade contract is not ex-post efficient since a contract such as z', where $z'_1(h)=(-1, 12)$ dominates the endowment for both agents in state h. However, the event $\{h\}$ is not common knowledge for the grand coalition, and it is clear that z' should be unacceptable to consumer 2. Indeed, any notion of the core that in this simple example leads to a core that is not Z^* must violate the common knowledge hypothesis. It would also be at odds with what is well understood to be the natural outcome in the lemons model.

[12] Example 1 of Wilson [38] illustrates this point with a single commodity but would not be adequate for the discussion that follows. In our two-commodity example, both z^*, the no-trade contract, and z are measurable with respect to the information partition of both agents, as required by the definitions in [16, 21, 39].

The definition of the core used by Yannelis [39] and Ichiishi and Idzik [21] (both in their pooling and non-pooling case) requires that *all* types of consumers in the objecting coalition gain.[13] It is not enough for a coalition to be able to object over an event that is discernible by it. Their definition would include z in the core because *both types of agent* 1 cannot improve upon it—even though agent 1 is fully informed, and can do better with her endowment if the state is h. This illustrates the consequence of not permitting a coalition to object over an event that is known to it.

Hahn and Yannelis [16] use a definition (interim private core) in which a coalition is permitted to object in any state t provided all members of the coalition can gain in terms of their conditional expected utility, given their respective individual information sets consistent with t. Agents in an objecting coalition are allowed to use different events over which they gain. According to their definition the grand coalition objects to $z^* \in Z^*$ with the contract z in state l because it provides higher utility to consumer 1 in the event $P_1(t) = \{l\}$, and to agent 2 in the event $P_2 = \{l, h\}$. However, z is not in their core either because it is objected to by consumer 1 in state h with the no-trade contract. In fact, the core, as defined by them is empty in this example. This illustrates the importance of requiring all agents to coordinate on a common event which is commonly known to them.

Our discussion of Example 2.1 makes a compelling case for allowing a coalition to object over an event commonly known to all its members. It also makes the point that allowing a coalition to objection over an event that is not a common knowledge event may be unreasonable; such an objection may inform the uninformed agent that the true state is in fact one in which she would be worse-off. To be sure, this does not mean that an objection over an event that is not discernible is unviable. But it does suggest that any such objection needs to be scrutinized more carefully before it can be judged viable. Our definition of the core takes the simplistic view that common knowledge of the event is not only sufficient but also necessary for an objecting coalition. It is possible that a coalition may have a "credible" objection even over an event that is not common knowledge to all members of the coalition. Note, however, that by incorporating such considerations, objections would be made easier, and any corresponding notion of the core would be a refinement of the one that we have defined. Our main result is, therefore, robust to such considerations.

[13] Yannelies [39] concerns the more general model with a continuum of states and requires the conditional expected utility of each agent in an objecting coalition to be higher in almost all states.

3. NON-EMPTINESS OF THE CORE

Our main result consists of an example to show that the incentive compatible core may be empty. In order to clarify the crucial features of such an example we begin by describing a few special cases in which the core is non-empty.

In a model in which the information state is publicly known at the time contracts are delivered upon, the coarse core is non-empty under standard assumptions on preferences. Wilson [38] proved this result by showing that the corresponding characteristic function game is balanced and then applying Scarf's [36] theorem.[14]

Unfortunately, Wilson's proof cannot be extended to the case in which all contracts are required to be incentive compatible. This is due simply to the fact (observed by Prescott and Townsend [34], Hammond [18] and Allen [2]) that the set of incentive compatible allocations need not be convex; see in particular Figure 1 in [18]. It is easy to see that this problem can be ruled out by assuming that all indifference curves are linear; more precisely, by requiring all utility functions u_i to be affine linear. In this special case,[15] Wilson's argument can indeed be extended. We record this observation as the following remark.

Remark 3.1. Suppose $u_i(\cdot, t_i)$ is affine linear for all i and $t_i \in T_i$. Then the incentive compatible core is non-empty.

Remark 3.2. In a two-consumer economy, the incentive compatible core is simply the set of interim individually rational and incentive efficient contracts and is, therefore, nonempty. This observation follows from the fact that the set of interim individually rational and incentive compatible contracts is non-empty and compact, and the utility functions are continuous.

Another special case in which the incentive compatible core is non-empty is the one in which no agent has exclusive information. This kind of information structure was formalized by Postlewaite and Schmeidler [33] as follows.

Information is *non-exclusive* if for every $i \in N$ and $t \in T^*$,

$$\bigcap_{j \neq i} P_j(t) = \{t\}.$$

[14] He also pointed out (footnote 6, [38]) that an alternative proof can be provided by using the fact that the coarse core contains the equilibrium allocations of a "constrained market process." Interestingly, this equilibrium can be interpreted as a sunspot equilibrium in the restricted market participation economy of Cass and Shell [8]. See Theorem 5.7 in Goenka and Shell [14] for a similar connection in that context.

[15] See also Allen [2, 4], Ichiishi and Idzik [21] and the discussion in Section 4.

This is equivalent to the condition that

$$\text{for every } t \in T^*,\ q_i(t_{-i}, s_i) = 0 \qquad \text{for all} \quad i \in N \quad \text{and} \quad s_i \neq t_i$$

In other words, when information is non-exclusive any unilateral deception can be detected. In such environments incentive compatibility is easy to satisfy, especially for contracts that are interim individually rational, as the following lemma shows.

LEMMA 3.1. *Suppose information is non-exclusive and $x \in A_N$ is interim individually rational. Then there exists $\tilde{x} \in A_N$ such that $\tilde{x}$ is incentive compatible and equivalent to x in terms of interim utilities in the sense that*

$$U_i(\tilde{x}_i \mid t_i) = U_i(x_i \mid t_i) \qquad \textit{for all} \quad i \in N, \quad \textit{for all} \quad t \in T$$

Proof. See the Appendix.

Now consider any x in Wilson's coarse core. Since x is interim individually rational, by Lemma 3.1 there exists an incentive compatible allocation $\tilde{x}$ which is equivalent to x in terms of interim utility. Clearly, then $\tilde{x}$ belongs to the incentive compatible core, and we have (under the usual assumptions, namely continuity and concavity of utility functions):

PROPOSITION 3.1. *If information is non-exclusive, then the incentive compatible core is non-empty.*

While non-exclusiveness of information is a strong assumption, there are economically interesting cases in which it does hold, as our next Example illustrates.[16]

EXAMPLE 3.1. Modify Example 2.1 so that the seller has two indivisible units of an object and there are two buyers, each interested in buying at most one unit of the object. In state l both units are of low quality and in state h both units are of high quality. Suppose buyer 2 is fully informed. Thus the seller and buyer 2 are both of type l or type h. This makes the information structure non-exclusive. The valuation of the seller and the two buyers are given in the following table.

	seller	buyer 1	buyer 2
low	0	0	0
high	10	15	10

[16] I am grateful to Kenneth Arrow and Bill Zame for comments that led to this example and the following Remarks.

The coarse core (without incentive compatibility constraints) consists of trades in state h in which the buyer receives at least \$10 for each unit sold, buyer 1 pays between \$10 and \$15 and buyer 2 either does not trade or buys one unit for \$10. By Lemma 3.1 and Proposition 3.1 any such allocation can be modified to ensure that it is incentive compatible, and belongs to the incentive compatible core.[17] Note that in any such allocation the informed buyer cannot receive a positive surplus. It is interesting to observe that the incentive compatible core contains other allocations as well, which provide buyer 2 with a positive surplus. Consider, for example, a contract in which no trade takes place unless both the seller and buyer 2 report h. When they both report h, the trade is as follows. Buyer 1 pays \$15, buyer 2 pays \$9 and the seller delivers one unit to each buyer for a total of \$24. This cannot be objected to by the coalition consisting of the seller and buyer 1 since they cannot on their own capture the surplus while satisfying incentive compatibility constraints. Thus the incentive compatible core, in contrast to the coarse core, allows the informed buyer to obtain an informational rent.

Remark 3.3. This discussion also clarifies the importance of condition (ii) in the definition of a feasible contract for a coalition. The fact that the coalition consisting of the seller and buyer 1 cannot obtain a positive surplus (for the usual reasons, as in Example 2.1) is reflected in condition (ii) in the definition of A_S, which requires that this coalition *not* rely on messages from buyer 2.

Remark 3.4. According to Lemma 3.1 and Proposition 3.1, non-emptiness and ex-post efficiency of the incentive compatible core hold in the lemons economy as long as there are least two agents who are fully informed. This, for instance, would be the case in any replication of the two-agent lemons economy, where replication involves introducing sellers all of whom are fully informed.[18] Of course, a replication in which new sellers are only informed about the object they possess would introduce new states with each replication, and information may then not be non-exclusive. As Gul and Postlewaite [15] point out, that form of replication does not eliminate ex-post inefficiency in the lemons model even asymptotically.

While it may be possible to identify other interesting special cases in which the incentive compatible core is non-empty, our main result shows that a general result is not to be had. Clearly, this result also applies to any refinement of the incentive compatible core.

[17] The modification entails specifying a contract contingent on the states reported by both the informed agents, and requiring no-trade unless both informed agents report h.

[18] Lee [27] provides an extensive analysis of such a model.

PROPOSITION 3.2. *There exist well-behaved economies in which the incentive compatible core is empty.*

In a model in which contracts are made ex-ante, Allen [2] constructs an example with the aim to show that the incentive compatible core may be empty. Her example, however, is one with non-exclusive information and by Proposition 3.1, therefore, it cannot suffice for the purpose at hand.[19] To prove Proposition 3.2 we shall construct an example in which the incentive compatible core is empty. In light of the above discussion, our example must have at least three consumers and two states, an exclusive information structure and non-linear utility functions.

EXAMPLE 3.2. Consider an economy with three consumers, three commodities and two states of the world s and t. The true state of the world is known only to consumer 1. In other words, consumer 1 can be of type s and t, i.e., $T_1 = \{s, t\}$. Consumers 2 and 3 are of one type each. To save on notation, we will suppress the (unique) type of consumers 2 and 3 and let $T = T_1$. Both states are considered equally likely by consumers 2 and 3, i.e., $q_2(s) = q_3(s) = q_2(t) = q_3(t) = 0.5$. The consumption sets are R^3_+ in each state. Consumer i's endowment consists of commodity i in each state. The state independent endowments are

$$\omega_1(s) = \omega_1(t) = (16, 0, 0)$$

$$\omega_2(s) = \omega_2(t) = (0, 16.5, 0)$$

$$\omega_3(s) = \omega_3(t) = (0, 0, 16.5).$$

The state contingent utility functions are

$$u_1(x(s), s) = 4x_{11}(s) + 4x_{12}(s) + 4x_{13}(s)$$

$$u_1(x(t), t) = \sqrt{x_{11}(t)} + \sqrt{x_{12}(t)} + \sqrt{x_{13}(t)}$$

$$u_2(x(s), s) = \tfrac{9}{8}x_{21}(s) + 2\min[x_{22}(s), x_{23}(s)]$$

$$u_2(x(t), t) = 2x_{22}(t) + 2x_{23}(t)$$

$$u_3(x(s), s) = \tfrac{9}{8}x_{31}(s) + 2\min[x_{32}(s), x_{33}(s)]$$

$$u_3(x(t), t) = 2x_{32}(t) + 2x_{33}(t).$$

For a contract x, let

$$U_{23}(x) = u_2(x(s), s) + u_2(x(t), t) + u_3(x(s), s) + u_3(x(t), t).$$

[19] We discuss her example further in Section 4.

Notice that coalition $\{2, 3\}$, by sharing its endowment equally in both states, has access to an incentive compatible contract $\tilde{x}$ such that $U_{23}(\tilde{x}) = 99$.

To prove that the core is empty, we establish the following two claims.

CLAIM 1. *Suppose x is a contract with the following properties*:

(i) $u_1(x(s), s) \leqslant 64, \qquad u_1(x(t), t) \leqslant 4,$

(ii) $U_{23}(x) \leqslant 99.$

Then there is an objection to x from either coalition $\{1, 2\}$ *or* $\{1, 3\}$.

Proof of Claim 1. Suppose, without loss of generality, that $u_2(x(s), s) + u_2(x(t), t) \leqslant 49.5$. Now consider the following, feasible, contract for consumers 1 and 2.

$$\bar{x}_1(s) = (0, 16.5, 0), \qquad \bar{x}_1(t) = (16, 0.25, 0)$$

$$\bar{x}_2(s) = (16, 0, 0), \qquad \bar{x}_2(t) = (0, 16.25, 0).$$

Note that

$$u_1(\bar{x}(s), s) = 66, \qquad u_1(\bar{x}(t), t) = 4.5$$

$$u_2(\bar{x}(s), s) + u_2(\bar{x}(t), t) = 50.5$$

Thus both consumers 1 and 2 prefer $\bar{x}$ to x. It is easy to see that $\bar{x}$ is incentive compatible. Thus, $\bar{x}$ is an objection by coalition $\{1, 2\}$ to x. ∎

CLAIM 2. *The only interim individually rational, incentive compatible allocations which are immune to an objection from consumers 2 and 3 involve no trade with consumer 1.*

Clearly, these claims together imply that the core is empty.[20] It remains now to prove Claim 2.

The intuition for Claim 2 is the following. Consider a contract such that $x_1(s) = (0, 8, 8)$. Consumers 2 and 3 make a total surplus of 2 units. They can, therefore, give up no more than 1 unit in state t. The maximum $u_1(t)$ that this allows is through $x_1(t) = (16, 0.5, 0.5)$ and $u_1(t) \leqslant \sqrt{16} + \sqrt{0.5} + \sqrt{0.5} = 4 + \sqrt{2} = 5.4$. However, $u_1(x_1(s), t) = 2\sqrt{8} = \sqrt{32} = 5.65$ which violates incentive compatibility in state t. The proof of Claim 2, of course, has to deal with other kinds of contracts as well.

[20] Claims 1 and 2 can also be used to show that the underlying characteristic function game is not balanced.

The following Lemma, based only on the specification of the economy, will be useful. For notational convenience, we use U_{23} to refer to $U_{23}(x)$.

LEMMA 3.2. *If x is a feasible allocation, then*

$$2\max[x_{12}(s), x_{13}(s)] \leqslant 117 - U_{23} \tag{2}$$

$$2[x_{12}(t) + x_{13}(t)] \leqslant 117 - U_{23} - \tfrac{1}{4}u_1(x(s), s) - \tfrac{1}{8}x_{11}(s) \tag{3}$$

$$u_1(x(t), t) \leqslant 4 + \sqrt{117 - U_{23} - \tfrac{1}{4}u_1(x(s), s) - \tfrac{1}{8}x_{11}(s)}. \tag{4}$$

Proof. See the Appendix.

Proof of Claim 2. Consider an individually rational, incentive compatible contract x such that consumers 2 and 3 do not have an objection to x, i.e., $U_{23} \geqslant 99$. Individual rationality for agent 1 in state s implies that $u_1(x(s), s) \geqslant 64$, or

$$x_{11}(s) \geqslant 16 - x_{12}(s) - x_{13}(s). \tag{5}$$

Conditions (2) and (4) of Lemma 3.2, now simplify to

$$\max[x_{12}(s), x_{13}(s)] \leqslant 9 \tag{6}$$

$$u_1(x(t), t) \leqslant 4 + \sqrt{2 - \tfrac{1}{8}x_{11}(s)}. \tag{7}$$

Incentive compatibility for consumer 1 requires that

$$u_1(x(t), t) \geqslant \sqrt{x_{11}(s)} + \sqrt{x_{12}(s)} + \sqrt{x_{13}(s)}.$$

Using (7), this means that

$$4 + \frac{1}{\sqrt{8}}\sqrt{16 - x_{11}(s)} \geqslant \sqrt{x_{11}(s)} + \sqrt{x_{12}(s)} + \sqrt{x_{13}(s)}. \tag{8}$$

Next, we will show that $x_{11}(s) \geqslant 7$. Suppose not, i.e., $x_{11}(s) \in [0, 7)$. Notice that while $\sqrt{x_{12}(s)} + \sqrt{x_{13}(s)}$ is minimized by making the two terms as unequal as possible, (6) provides a restriction on how far these terms can diverge. From (5) and (6), it follows that

$$\sqrt{x_{12}(s)} + \sqrt{x_{13}(s)} \geqslant \sqrt{9} + \sqrt{7 - x_{11}(s)}.$$

Notice the left hand side of (8) is bounded above by $4 + \sqrt{2} = 5.14$. Thus, (8) implies that

$$5.14 \geqslant 3 + \sqrt{x_{11}(s)} + \sqrt{7 - x_{11}(s)}. \tag{9}$$

Of course, $\sqrt{x_{11}(s)}+\sqrt{7-x_{11}(s)} \geqslant \sqrt{7}=2.64$. But this contradicts (9) and proves that $x_{11}(s) \in [7, 16]$.

Since

$$\sqrt{x_{12}(s)}+\sqrt{x_{13}(s)} \geqslant \sqrt{x_{12}(s)+x_{13}(s)} \geqslant \sqrt{16-x_{11}(s)},$$

(8) implies that

$$4+\frac{1}{\sqrt{8}}\sqrt{16-x_{11}(s)} \geqslant \sqrt{x_{11}(s)}+\sqrt{16-x_{11}(s)}$$

or,

$$h(x_{11}(s))=4+\left(\frac{1}{\sqrt{8}}-1\right)\sqrt{16-x_{11}(s)}-\sqrt{x_{11}(s)} \geqslant 0.$$

Notice that $h(16)=0$. The function $h(x_{11}(s))$ is U-shaped, with a positive derivative at 16. The derivative of h is negative at $x_{11}(s)=7$ but the value there is

$$h(7)=4+\left(\frac{1}{\sqrt{8}}-1\right)3-\sqrt{7}=4-1.94-2.64<0.$$

Thus $h(x_{11}(s))<0$ for all $x_{11}(s) \in (7, 16)$. Since incentive compatibility requires that $h(x_{11}(s)) \geqslant 0$, and this implies that $x_{11}(s)=16$, there cannot be any trade with consumer 1. This completes the proof of Claim 2. ▌

4. EX-ANTE CONTRACTS

Suppose contracts are executed at the ex-ante stage, i.e., before any private information is revealed, and that agents receive private information before the maturity data. An appropriate notion of the core in the ex-ante context will focus on ex-ante utilities.[21] The only modification that this requires to the definitions in Section 2 is to modify condition (1) in defining objections to

$$\sum_{t \in T} q_i(t)\, u_i((y_i(t), t) > \sum_{t \in T} q_i(t)\, u_i((x_i(t), t) \qquad \text{for all} \quad i \in S.$$

The distinction between this framework and the one we have used so far is analogous to that between ex-ante efficiency and interim efficiency.

[21] Additional simplicity follows from the fact that the only comment knowledge event, ex-ante, is T.

However, there is one important respect in which this analogy does not hold. While ex-ante (incentive) efficient contracts are interim (incentive) efficient (see [20, 30]) there is no logical relationship between the ex-ante core and the interim core. This stems, again, from the fact that a coalition which is a strict subset of N may be able to improve upon the status-quo over an event E but not over the complement of E,[22] and not in ex-ante terms. Indeed, the interim core can be disjoint from the ex-ante core; in Example 2.1, the interim core is Z^* while the ex-ante core (containing contracts such as z) is disjoint from Z^*. Results on the emptiness or the non-emptiness of the core must therefore be proved separately for the ex-ante and the interim versions.

Remarks 3.1 and 3.2 remain valid for the ex-ante incentive compatible core.

Proposition 3.1 makes use of interim individual rationality to ensure that the modified contract satisfies exact feasibility. However, it is easy to see that the argument used in proving Proposition 3.1 does extend to the ex-ante framework provided we define $\tilde{x}(t)$ in the proof of Lemma 3.1 to be 0 when $t \notin T^*$ (assume free disposal in states that occur with 0 probability) and assume that $u_i(x(t), t) \geqslant u_i(0, t)$ for all $i \in N$ and $x \in A_N$. It is also easy to see, by the usual balancedness arguments, that the ex-ante coarse core (without incentive constraints) is non-empty. Putting these observations together, we have:

PROPOSITION 4.1. *The ex-ante coarse core is non-empty. Suppose private information is non-exclusive, and $u_i(x(t), t) \geqslant u_i(0, t)$ for all $i \in N$ and $x \in A_N$. If x belongs to the ex-ante coarse core, then $\tilde{x}$ belongs to the ex-ante, incentive compatible core, where*

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if} \quad t \in T^* \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1 is motivated by a three-consumer, five-state example of Allen [2] in which she claims that the ex-ante, incentive compatible core is empty. For the present discussion it will suffice to consider the following partial description of Allen's example. The five states are denoted (a, b, c, d, f). It is assumed that agent 1 can distinguish between all states. Agents 2 and 3 can distinguish between states c, d and f. When the state is a or

[22] Recall from our discussion following the definition of the core in Section 2, that this is also the reason that a subcoalition of N need not improve the interim utility for all types. If, as in Yannelis [39] or Ichiishi and Idzik [21], interim objections are required to make all types better off, then the ex-ante core is indeed a subset of the interim core; Ichiishi and Idzik [21] actually prove non-emptiness of the ex-ante core, and deduce from it the non-emptiness of their interim core.

b agents 2 and 3 receive a signal $s_i \in \{0, 1\}$. When the state is a they both receive the same signal; and the signal is 0 or 1 with probability 0.5 each. If the state is b they receive the signals $(0, 1)$ or $(1, 0)$ with equal probability. Clearly then, information in this example is non-exclusive. From Proposition 4.1, therefore, it follows that the ex-ante, incentive compatible core is non-empty if we allow for free disposal in any state $t \notin T^*$.[23] It is natural to ask whether an example (with exclusive information) can be constructed in which the ex-ante, incentive compatible core is empty. Indeed, Example 3.2 serves this purpose and yields our next result.

PROPOSITION 4.2. *There exist well-behaved economies in which the ex-ante, incentive compatible core is empty.*

Proof. See the Appendix.

Remark 4.1. An interesting, alternative approach is to consider a model with random allocations. The case for doing so has recently been well laid out in Forges and Minelli [13], where it is shown that the economy of Example 3.2 above does indeed possess a non-empty ex-ante, incentive compatible core if random allocations are allowed. See also the Proposition in [13]. However, a general, positive result remains elusive. Thus, whether Propositions 3.2 and 4.2 apply to a model with random allocations remains an important open issue for future research.

5. MEASURABILITY OF CONTRACTS

An alternative framework, quite distinct from ours, is one in which the agents *cannot* communicate through a mediator, and contracts are required to be contingent only on the information available to a coalition at the time of signing a contract. If there is no information pooling this translates into the condition that each agent's net trade be measurable with respect to the agent's private information. This is approach is related to the literature which takes the view that the amount of information shared within a coalition corresponds to an appropriate measurability restriction on the contracts; see for example, Allen [3, 5], Ichiishi and Idzik [21], Hahn and Yannelis [17], Koutsougeras and Yannelis [26], and Yannelis [39].[24]

[23] In fact, free disposal is implicitly assumed by Allen [2] since her definition of Bayesian incentive compatibility imposes zero consumption when $t \notin T^*$. Moreover, in her example, it can be shown that the ex-ante core is non-empty even if free disposal is not allowed.

[24] An analogy with Radner's [35] approach requires some care. Radner argues that an agent's *act* at a certain date be measurable with respect to the agent's private information at that date. He does not require that a contract for future delivery be measurable with respect to current information.

Clearly, measurability can be a severe restriction on the opportunities for insurance. In fact, in an exchange economy a measurable net-trade is a fortiori incentive compatible. Consider the simple case in which $T=T^*$.[25]

A net-trade z is *measurable with respect to the private information of agent i* if i's net-trade is the same in states that are indistinguishable by i, i.e., if $z_i(t)=z_i(\bar{t})$ for all t, $\bar{t}\in P_i(t_i)$. When $T=T^*$ this means that

$$z_i(t)=z_i(\bar{t}) \qquad \text{for all} \quad t,\ \bar{t}\in T \quad \text{such that} \quad t_i=\bar{t}_i. \tag{10}$$

A net-trade is said to be *measurable* if it is measurable with respect to the private information of each agent. Given that $\omega_i(t)$ depends only on t_i, i.e., ω is measurable, measurability of x is equivalent to the measurability of the corresponding net-trade $z(x)$.

PROPOSITION 5.1. *Suppose $T=T^*$ and $x\in A_N$ is a measurable contract. Then the net-trade corresponding to x is independent of t, and x is therefore incentive compatible.*

Proof. See the Appendix.

An immediate corollary of this Proposition is that in a single commodity economy (with monotonic preferences), the only measurable, individually rational contract is no-trade. This is related to Proposition 5.2 of Koutsougeras and Yannelis [26]; see also Section 8 in Radner [35] and Proposition 4.2 in Allen [3]. Ichiishi and Radner [22, Lemma 6.3] show that if prefernces are monotonic, the conclusion of Proposition 5.1 applies even to measurable contracts satisfying the resource constraint with a weak inequality.

The rationale given by Hahn and Yannelis [17] and Allen [5] for making this measurability assumption is that if an allocation is not measurable with respect to private information it may create incentive problems. However, imposing measurability is an unnecessarily strong way of ensuring incentive compatibility. To see this, modify Example 2.1 such that buyer assigns a value of 2 to the object in the low state. Then the incentive compatibility core consists only of those contracts that involve trade in state l—the standard prediction in the lemons problem. But such a contract is not measurable with respect to the information of the buyer. Indeed, in this example, the no-trade contract is the *only* measurable and feasible contract satisfying (interim) individual rationality.

The measurability restriction, by rendering trivial the incentive compatibility issue, also makes it easy to establish non-emptiness of the core.

[25] This is the approach followed by Ichiishi and Idzik [21]. Their model, however, is more general and includes exchange economies as one particular case.

Since a convex combination of measurable trades is measurable, non-emptiness of the core (ex-ante or interim) follows, at least in a finite economy with a finite number of types, from the usual arguments.[26] It is not necessary, therefore, to impose affine linearity of the utility functionbs as is done in Example 5.7 of Ichiishi and Idzik [21]. It is in the absence of the measurability assumption that Propositions 3.2 and 4.2 become relevant.

APPENDIX

Proof of Lemma 3.1. Let $x \in A^N$ be interim individually rational. Define

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if} \quad t \in T^* \\ \omega(t) & \text{otherwise.} \end{cases}$$

Since $u_i(\tilde{x}_i, t) = u_i(x, t)$ for all $t \in T^*$, it follows that $\tilde{x}$ is equivalent to x in terms of interim utility for every $i \in N$ and every $t \in T$.

To complete the proof we shall now show that $\tilde{x}$ is incentive compatible. Consider $s_i \in T_i$ such that $s_i \neq t_i$.

$$\begin{aligned} U_i(\tilde{x}_i, s_i \mid t_i) &= \sum_{t_{-i} \in T_{-i}} q_i(t_{-i} \mid t_i)\, u_i(z_i(\tilde{x}(t_{-i}, s_i)) + \omega_i(t_i)), t) \\ &= \sum_{\{t_{-i} \in T_{-i} \mid (t_{-i},\, t_i) \in T^*\}} q_i(t_{-i} \mid t_i)\, u_i(z_i(\tilde{x}(t_{-i}, s_i)) + \omega_i(t_i)), t). \end{aligned} \tag{11}$$

By non-exclusivity, for every $(t_{-i}, t_i) \in T^*$ we know that $(t_{-i}, s_i) \notin T^*$. From the construction of $\tilde{x}$ it follows that $z(\tilde{x}(t_{-i}, s_i)) = 0$ for all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$. Thus (11) can be rewritten as

$$\begin{aligned} U_i(\tilde{x}_i, s_i \mid t_i) = &\sum_{\{t_{-i} \in T_{-i} \mid (t_{-i},\, t_i) \in T^*\}} q_i(t_{-i} \mid t_i)\, u_i(\omega_i(t_i), t) \\ &\leqslant U_i(x_i \mid t_i) = U_i(\tilde{x}_i \mid t_i), \end{aligned}$$

where the inequality follows from interim individual rationality of x. Thus $\tilde{x}$ is incentive compatible. ∎

Proof of Lemma 3.2. Given the utility functions of consumers 2 and 3,

$$\begin{aligned} &\tfrac{9}{8}[16 - x_{11}(s)] \\ &\quad + 2\min[16.5 - x_{12}(s), 16.5 - x_{13}(s)] + 2[33 - x_{12}(t) - x_{13}(t)] \geqslant U_{23} \end{aligned}$$

[26] Yannelis [39] shows non-emptiness for economies in which the commodity space is a separable Banach lattice.

which imposes the following bound on $2[x_{12}(t)+x_{13}(t)]$

$$2[x_{12}(t)+x_{13}(t)] \leqslant 117-U_{23}-2\max[x_{12}(s), x_{13}(s)]-\tfrac{9}{8}x_{11}(s) \quad (12)$$

and immediately yields (2).

Notice that

$$x_{11}(s)=\tfrac{1}{4}u_1(x(s), s)-x_{12}(s)-x_{13}(s). \quad (13)$$

Substituting this in (12) we obtain

$$2[x_{12}(t)+x_{13}(t)] \leqslant 117-U_{23}-2\max[x_{12}(s), x_{13}(s)]-\tfrac{1}{8}x_{11}(s) \\ -[\tfrac{1}{4}u_1(x(s), s)-x_{12}(s)-x_{13}(s)].$$

Since $x_{12}(s)+x_{13}(s)-2\max[x_{12}(s), x_{13}(s)] \leqslant 0$, this yields (3).

Clearly,

$$\sqrt{x_{12}(t)}+\sqrt{x_{13}(t)} \leqslant \sqrt{2[x_{12}(t)+x_{13}(t)]}. \quad (14)$$

Since

$$u_1(x(t), t)=\sqrt{x_{11}(t)}+\sqrt{x_{12}(t)}+\sqrt{x_{13}(t)},$$

(4) follows from (3) and (14).

Proof of Proposition 4.2. Consider Example 3.2. The only reason that Proposition 3.2 does not by itself furnish a proof is that it relied on interim individual rationality for consumer 1 in state s. In other words, the proof of Proposition 3.2 suffices to show that the ex-ante, incentive compatible core does not contain any contract in which $u_1(x(s), s) \geqslant 64$. We, therefore, only need to consider the possibility of a core allocation, x, in which

$$u_1(x(s), s)<64. \quad (15)$$

Of course, x must be immune to an objection from coalition $\{2, 3\}$, i.e., $U_{23} \geqslant 99$. We can use condition (4) of Lemma 3.2 to assert that

$$u_1(x(t), t) \leqslant 4+\sqrt{18-\tfrac{1}{4}u_1(x(s), s)}.$$

Combining this with (15), we have

$$u_1(x(s), s)+u_1(x(t), t) \leqslant 68+\sqrt{2}<70. \quad (16)$$

Recall from Claim 1 that consumers 1 and 2 (or 1 and 3) can obtain aggregate utilities 70.5 and 50.5 respectively. Since x is supposed to be in the core, this along with (16) must mean that $U_{23} \geqslant 101$. Substituting this in (4) yields

$$u_1(x(t), t) \leqslant 4 + \sqrt{16 - \tfrac{1}{4}u_1(x(s), s) - \tfrac{1}{8}x_{11}(s)}. \tag{17}$$

Since individual rationality for consumer 1 requires that $u_1(x(s), s) + u_1(x(t), t) \geqslant 68$, this implies that

$$u_1(x(s), s) + 4 + \sqrt{16 - \tfrac{1}{4}u_1(x(s), s)} \geqslant 68$$

or,

$$\sqrt{64 - u_1(x(s), s)} \geqslant 2[64 - u_1(x(s), s)]$$

which means that

$$u_1(x(s), s) \geqslant 63.75. \tag{18}$$

Substituting in (17) we also have

$$u_1(x(t), t) \leqslant 4.25 \tag{19}$$

and

$$x_{11}(s) \leqslant 0.5. \tag{20}$$

To complete the proof, we shall now show that (18), (19), and (20) contradict the incentive compatibility constraints for consumer 1. Incentive compatibility for consumer 1 requires that $u_1(x(t), t) \geqslant u_1(x(s), t)$. By (19), this means that

$$4.25 \geqslant \sqrt{x_{11}(s)} + \sqrt{x_{12}(s)} + \sqrt{x_{13}(s)}. \tag{21}$$

From (18) and (20) it follows that

$$x_{12}(s) + x_{13}(s) \geqslant 15.$$

We know from (2) that $x_{12}(s) \leqslant 9$ and $x_{13}(s) \leqslant 9$. Thus,

$$\sqrt{x_{12}(s)} + \sqrt{x_{13}(s)} \geqslant 3 + \sqrt{6}.$$

But now (21) implies that

$$4.25 \geqslant 3 + \sqrt{6}$$

which is an absurdity. ∎

Proof of Proposition 5.1. Consider a measurable contract x and $t \in T$ and $s_i \in T_i$. Let $z = z(x)$. Since $z_i(t') = -\sum_{j \in N;\, j \neq i} z_j(t')$ for all $i \in N$ and $t' \in T$,

$$z_i(t_{-i}, s_i) = -\sum_{j \in N;\, j \neq i} z_j(t_{-i}, s_i). \tag{22}$$

Since z is measurable, by (10), $z_j(t_{-i}, s_i) = z_j(t)$ for all $j \neq i$. Substituting this in (22) we have

$$z_i(t) = z_i(t_{-i}, s_i).$$

Since this holds for all $i \in N$, $t \in T$ and $s_i \in T_i$,

$$z_i(t) = z_i(t') \qquad \text{for all} \quad t, t' \in T, \quad \text{for all} \quad i \in N. \quad \blacksquare$$

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